

Linear-Quadratic Approximation, External Habit and Targeting Rules*

Paul Levine
University of Surrey

Joseph Pearlman
London Metropolitan University

Richard Pierse
University of Surrey

November 30, 2006

Abstract

We examine the linear-quadratic (LQ) approximation of non-linear stochastic dynamic optimization problems in macroeconomics, in particular for monetary policy. We make four main contributions: first, we draw attention to a general Hamiltonian framework for LQ approximation due to Magill (1977). We show that the procedure for the ‘large distortions’ case of Benigno and Woodford (2003, 2005) is equivalent to the Hamiltonian approach, but the latter is far easier to implement. Second, we apply the Hamiltonian approach to a Dynamic Stochastic General Equilibrium model with external habit in consumption. Third, we introduce the concept of target-implementability which fits in with the general notion of targeting rules proposed by Svensson (2003, 2005). We derive sufficient conditions for the LQ approximation to have this property in the vicinity of a zero-inflation steady state. Finally, we extend the Hamiltonian approach to a non-cooperative equilibrium in a two-country model.

JEL Classification: E52, E37, E58

Keywords: Linear-quadratic approximation, dynamic stochastic general equilibrium models, utility-based loss function.

*We acknowledge financial support for this research from the ESRC, project no. RES-000-23-1126 and from the European Central Bank’s Research Visitors Programme for Paul Levine. Thanks are owing to the ECB for this hospitality and to numerous resident and visiting researchers for stimulating discussions. Views expressed in this paper do not necessarily reflect those of the ECB. An earlier version of this paper was presented at the 12th conference on Computing in Economics and Finance in Limassol, Cyprus, June 22-24, 2006; at the 4th Centre for Dynamic Macroeconomic Analysis Conference, University of St Andrews, Sept 6-8, 2006 and at seminars at the Bank of England and the University of Surrey. We have benefited from comments of participants at all these events and from further discussions with Junill Kim.

Contents

1	Introduction	1
2	The LQ Approximation to a General Dynamic Optimization Problem	3
2.1	Necessary and Sufficient Conditions for an Accurate LQ Approximation: . . .	3
2.2	A Simple Example	6
3	The Social Planner's and Ramsey Problems	7
3.1	Characterization of the Efficient Level	8
3.2	The Flexible-Price Solution and the Ramsey Problem	8
3.3	LQ Approximation of the Ramsey Problem: Efficient Case	10
3.4	The Small Distortion Case	11
4	LQ Approximation of Optimal Monetary Policy in a DSGE Model	12
4.1	Price Dispersion and the Costs of Inflation	15
4.2	The Ramsey Problem	16
4.3	Linearization of Dynamics	17
4.4	The Accurate LQ Approximation	18
4.5	The Social Planner's Problem	18
4.6	The Efficient Case	20
4.7	The Small Distortion Case	20
5	Target Implementability and External Habit	21
6	Nash Equilibria in a Two-Bloc Model	24
6.1	Non-cooperative Social Planners' Problem	24
6.2	Choice of Subsidy to Match the Efficient Level	25
6.3	LQ Approximation for the Small Distortions Case	25
6.4	LQ Approximation for the Large Distortions Case	27
7	Concluding Remarks	30
A	The Equivalence of the Benigno-Woodford and Hamiltonian Procedures	31
A.1	The Benigno-Woodford Procedure	31
A.2	The Hamiltonian Procedure	32
B	Proof of Proposition	32
C	Derivation of Second-Order Welfare Approximation in the Two-Bloc Case:	33

1 Introduction

This paper is about macroeconomic optimization problems in general with a particular application to welfare-optimal monetary policy. The title may sound like a smorgasbord of disparate topics, but in fact, as we will demonstrate, the correct linear-quadratic (LQ) approximation of a non-linear optimization problem, external habit in DSGE models and the idea of general targeting monetary rules as proposed by Svensson (2003) and Svensson (2005) are all related.

LQ approximations to non-linear dynamic optimization problems in macroeconomics are widely used for a number of reasons. First, for LQ problems the characterization of time consistent and commitment equilibria for a single policy maker, and even more so for many interacting policymakers, are well-understood. Second, the certainty equivalence property results in optimal rules that robust in the sense that they are independent of the variance-covariance matrix of additive disturbances. Third, policy can be decomposed into deterministic and stochastic components. This is a very convenient property since it enables the stochastic stabilization component to be pursued using simple Taylor-type feedback rules rather than the exceedingly complex optimal counterpart. Fourth, the stability of the system is conveniently summarized in terms of eigenvalues. Finally for sufficiently simple models, LQ approximation allows analytical rather than numerical solution.

But what is the correct procedure for replacing a stochastic non-linear optimization problem with a LQ approximation? Until quite recently some common methods employed by economists have produced poor approximations which open up the possibility of spurious results. These pitfalls are very neatly exposed in Kim and Kim (2003) and Kim and Kim (2006).

We make four main contributions to the optimal macroeconomic policy literature. First, we draw attention to a general Hamiltonian framework for approximating a non-linear problem by an LQ one due to Magill (1977), who appears to be the first to have applied it in the economics literature. We use a simple example to show that the procedure for the ‘large distortions’ case of Benigno and Woodford (2003, 2005) is equivalent to the Hamiltonian approach, but the latter is far easier to implement. Second, we apply the Hamiltonian approach to a fairly standard New Keynesian DSGE model with external habit in consumption. We show that in such a model the natural rate of output can exceed the efficient rate if habit is sufficiently strong and this feature has important implications for the form of the LQ approximation. Third, we introduce the concept of target-implementability which requires the quadratic approximation of the welfare to be

expressed in terms of targets or ‘bliss points’ for linear combinations of macroeconomic variables. Such a property fits in with the notion of targeting rules proposed by Svensson (2003, 2005). We then derive sufficient conditions for the LQ approximation to have this property in the vicinity of a zero-inflation steady state. Finally we extend the Hamiltonian approach to a non-cooperative equilibrium in a two-country model. In the simple model of Clarida *et al.* (2002) without habit in consumption, we show that the commonly used procedures for LQ approximation assuming either an efficient zero-inflation steady state (achieved by a subsidy) or ‘small distortions’ are inappropriate unless the instrument in the Nash equilibrium is assumed to be output. If, perhaps more plausibly, the instrument is taken to be inflation (or inflation targets perfectly achieved) then the Hamiltonian approach, which we extend to the two-country non-cooperative equilibrium, must be employed.

The rest of the paper is organized as follows. Section 2 summarizes the results of a discrete-time version of the continuous-time results of and Magill (1977). We provide a simple example of this, using a simple New Keynesian model and ad hoc policymaker’s utility function as set out in Clarida *et al.* (1999) and demonstrate that for this simple example the BW and Hamiltonian procedures are equivalent.

In Section 3 we introduce the general class of DSGE models to be studied. We examine the social planner’s problem where optimization is only subject to resource constraints. We then proceed to the Ramsey problem where, in addition, the policy-maker faces constraints in the form of decentralized decisions by households and firms, given the instrument at her disposal. The LQ approximation to the Ramsey problem is analyzed for the ‘efficient case’ (where subsidies eliminate distortions in the steady state) and the ‘small distortions case’ where such subsidies are not available.

Section 4 applies the general results of preceding sections to welfare-maximizing monetary policy in a fairly standard New Keynesian framework, with external habit in consumption. We derive the corresponding LQ approximation to the policymaker’s problem, and briefly comment on its representation when there is no habit. In addition we compare the quadratic expansions in the efficient case and the large distortions case.

Section 5 defines and discusses the notion of *target-implementability*; this is essentially about the conduction of monetary policy by the setting of targets in the loss function by the monetary authority, as advocated by Svensson, when it engages in stabilization policy. We then obtain sufficient conditions for the target implementability of the LQ approximation to the Ramsey problem in the vicinity of a zero-inflation steady state. Finally, to demonstrate that second-order conditions for a maximum do matter, we provide an example for which the necessary and sufficient second-order conditions for optimality

are not satisfied although the zero-inflation steady state satisfies the necessary first-order conditions.

In Section 6, we extend our work to the two bloc case. We demonstrate the inadequacy of the ‘small distortions’ approach, and derive an appropriate LQ approximation for the ‘large distortions’ case, which we compare with Benigno and Benigno (2006). Section 7 provides some concluding remarks.

2 The LQ Approximation to a General Dynamic Optimization Problem

In this section we provide the mathematical background to LQ approximations that provide accurate first-order approximations to deviations from the optimal solution to a deterministic dynamic optimization (or optimal control) problem. We firstly state the requirements for an LQ approximation to be accurate to first order, and then explain how it may be obtained using the Lagrangian of the problem. We apply this to a very simple economic model with a rudimentary output/inflation tradeoff. The reason for so doing is to demonstrate that the method of Benigno and Woodford (2003, 2005) as used in this model is equivalent to implementing a Lagrangian (or more strictly a Hamiltonian) approach to the objective function.

2.1 Necessary and Sufficient Conditions for an Accurate LQ Approximation:

Suppose we have a dynamic optimization problem expressed in the form¹

$$\max \sum_{t=0}^{\infty} \beta^t U(X_{t-1}, W_t) \quad s.t. \quad X_t = f(X_{t-1}, W_t) \quad (1)$$

given initial and possibly transversality conditions, which has a steady state solution \bar{X}, \bar{W} for the states X_t and the policies W_t . Define $x_t = X_t - \bar{X}$ and $w_t = W_t - \bar{W}$ as representing the first-order approximation to deviations of states and policies from their steady states. Suppose we write a candidate first-order approximation to the problem as

$$\max \sum_{t=0}^{\infty} \beta^t \left(\begin{bmatrix} x_{t-1}^T & w_t^T \end{bmatrix} Q \begin{bmatrix} x_{t-1} \\ w_t \end{bmatrix} - 2 \begin{bmatrix} x_{t-1}^T & w_t^T \end{bmatrix} b \right) \quad s.t. \quad x_t = Ax_{t-1} + Bw_t + c \quad (2)$$

¹An alternative representation of the problem is $U(X_t, W_t)$ and $E_t[X_{t+1}] = f(X_t, W_t)$ where X_t includes non-predetermined variables and $E_t[X_{t+1}] = X_{t+1}$ for the deterministic problem where perfect foresight applies. Whichever one uses, it is easy to switch from one to the other by a simple re-definition.

Then a necessary condition for this to be a first-order accurate solution is that $b = 0$, $c = 0$ i.e. the objective function must be purely quadratic in deviations, and the dynamics purely linear in deviations. The reasons for this are clear; firstly, suppose that the system starts in the steady state; it could only remain in the steady state $x_t = 0$, $w_t = 0$ if $c = 0$. Secondly, if $b \neq 0$ then there is a bliss-point $Q^{-1}b \neq 0$ which would be desirable, so that the solution to the problem starting at the steady state would not remain at the steady state.

The implications of these conditions are rather serious given the manner in which many LQ approximations were conducted within the economics profession up until a few years ago. Often, the approach used would often not even involve finding the steady state of the optimal solution to the problem. Even were that the case, the next error to be committed would be to use merely a Taylor series approximation to the objective function $f(X, W)$, whose first-order expansion cannot of course be guaranteed to equal zero.

However there exists a very straightforward approach to finding the appropriate approximation. Define the problem using a Lagrangian L

$$L = \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \lambda_t^T (X_t - f(X_{t-1}, W_t))] \quad (3)$$

so that a *necessary* condition for the solution to (1) is that the Lagrangian is stationary at all $\{X_s\}$, $\{W_s\}$ i.e.

$$U_W + \lambda_t^T f_W = 0 \quad U_X - \frac{1}{\beta} \lambda_{t-1}^T + \lambda_t^T f_X = 0 \quad (4)$$

These necessary conditions for an optimum do not imply that there is an asymptotic steady state to (4). However for the purposes of this paper, let us assume that this is the case, so that a steady state $\bar{\lambda}$ for the Lagrange multipliers exists as well. Now define the Hamiltonian $H_t = f(X_{t-1}, W_t) + \bar{\lambda}^T f(X_{t-1}, W_t)$. The following is the discrete time version of Magill (1977):

Theorem 1: If a steady state solution $(\bar{X}, \bar{W}, \bar{\lambda})$ to the optimization problem (1) exists, then any perturbation (x_t, w_t) about this steady state can be expressed as the solution to

$$\text{Max} \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} x_{t-1} & w_t \end{bmatrix} \begin{bmatrix} H_{XX} & H_{XW} \\ H_{WX} & H_{WW} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ w_t \end{bmatrix} \quad \text{s.t.} \quad x_t = f_X x_{t-1} + f_W w_t \quad (5)$$

where H_{XX} , etc denote second-order derivatives evaluated at (\bar{X}, \bar{W}) .

The reason why this result holds is because the derivatives of the Lagrangian with respect to X_t and W_t are zero when evaluated at $(\bar{X}, \bar{W}, \bar{\lambda})$. By definition, $\sum_{t=0}^{\infty} \beta^t U(X_{t-1}, W_t) = \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \bar{\lambda}^T (X_t - f(X_{t-1}, W_t))]$, and the first-order term of the Taylor

series expansion of the latter expression is zero.

For the result of theorem to hold $(\bar{X}, \bar{W}, \bar{\lambda})$ must satisfy (4). These, it should be stressed, are necessary but not sufficient conditions for a local maximum. A standard sufficient condition for optimality is that the functions $f(X, W)$ and $U(X, W)$ are concave, but this is rarely satisfied in examples from economics. A more useful sufficient condition is the following:

Theorem 2: A sufficient condition for the steady state of (4) to be a local maximum is that the matrix of second derivatives of H in (5) is negative semi-definite².

This condition is easy to check, but in the event that it does not hold, the following discrete time version of the sufficient conditions for an optimum Magill (1977) is applicable when the constraints and/or the welfare function are non-concave. It is based on iterating the quadratic approximation to the value function.

Theorem 3:

A necessary and sufficient condition for the solution (4) to the dynamic optimization problem (1) to be a local maximum is that $\beta f_W^T P_t f_W + H_{WW}$ is negative definite for all t , where the matrices $f_X, f_W, H_{XX}, H_{XW}, H_{WW}$ are all evaluated along the solution path and P_t satisfies the Riccati equation given by:

$$P_{t+1} = \beta f_X^T P_t f_X - (\beta f_X^T P_t f_W + H_{XW})(\beta f_W^T P_t f_W + H_{WW})^{-1}(\beta f_W^T P_t f_X + H_{WX}) + H_{WW} \quad (6)$$

and the value function of small perturbations x_t about the path of the optimal solution dynamic optimization problem is given by $\frac{1}{2}x_t^T P_t x_t$, with initial condition $P_0 = 0$.

As mentioned above we assume the existence of a steady state solution to (4) given by $[\bar{X}, \bar{W}, \bar{\lambda}]$, since we are interested in approximations about the latter. Hence the matrices in (6) (apart from P_t) are constant. Thus this theorem provides a means of checking whether a candidate solution to (4) actually is optimal. Note that the perturbed system is in standard linear-quadratic format, which is the basis for this result.

We also note that Magill (1977)'s result extends to the stochastic case as well. Thus if the dynamic equations are written as $X_t = f(X_{t-1}, W_t, \varepsilon_t)$, where the ε_t have mean zero and are independently normally distributed then any perturbations about the determin-

²A simple example of a problem for which a maximum exists, but for which this sufficient condition does not hold is: $\max x^2 - y^2$ such that $y = ax + b$. It is easy to see that the stationary point is a maximum when $|a| > 1$.

istic solution are solutions to the problem

$$\begin{aligned}
Max E_0 \quad & \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{ccc} x_{t-1} & w_t & \varepsilon_t \end{array} \right] \begin{bmatrix} H_{XX} & H_{XW} & H_{X\varepsilon} \\ H_{WX} & H_{WW} & H_{W\varepsilon} \\ H_{\varepsilon X} & H_{\varepsilon W} & H_{\varepsilon\varepsilon} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ w_t \\ \varepsilon_t \end{bmatrix} \\
s.t. \quad & x_t = f_X x_{t-1} + f_W w_t + f_\varepsilon \varepsilon_t
\end{aligned} \tag{7}$$

2.2 A Simple Example

We now present an application of this theorem to a simple stylized problem, which also summarizes the erroneous approach to LQ approximation. In addition, we outline how the method of Benigno and Woodford (2003, 2005) would be implemented to find the LQ approximation. The two methods are equivalent, but the Hamiltonian approach is both more general and more straightforward to implement.

Consider the following optimization problem for a monetary authority to choose a state-contingent path for its inflation target π_t so as to minimize an ad hoc objective function

$$E_0 \left[\sum_{t=0}^{\infty} \lambda^t [(x_t - x^*)^2 + \pi_t^2] \right] \tag{8}$$

where x_t is the output gap in logarithms given by a non-linear ‘New Keynesian’ Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + f(x_t) + s_t; \quad f' > 0, f'' < 0 \tag{9}$$

where s_t is an i.i.d. supply shock. Let x_n be the natural rate of output defined by $f(x_n) = 0$. Then in (8) $x^* \geq x_n$ is the logarithm of the efficient level of output where inefficiency arises from monopolistic competition in the output market.

A common procedure for reducing this to a LQ problem is to expand about the steady state x_n so that $f(x_t) \simeq f(x_n) + f'(x_n)(x_t - x_n) + \frac{1}{2} f''(x_n)(x_t - x_n)^2 = a(x_t - x_n) - b(x_t - x_n)^2$. Much of the literature³ including Clarida *et al.* (1999) then erroneously adopts a linearized Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + a(x_t - x_n) + s_t \tag{10}$$

and proceeds with the LQ problem of minimizing (8) subject to (10). The error arises from the objective function including a linear term in $x_t x^*$. From (9), $x_t = f^{-1}(\pi_t - \beta E_t \pi_{t+1} - s_t)$ so unless x^* is small, there is a second-order term missing in the objective function if one proceeds with the linear approximation (10).

To get round this problem, the procedure set out in BW considers the deterministic problem with commitment, so that there is no need to distinguish between $E_t \pi_{t+1}$ and

³Some previous work of one of the authors joins a distinguished list (see Currie and Levine (1993)).

π_{t+1} . Then one selects a new steady state $(\bar{\pi}, \bar{x})$ satisfying (9), and a multiplier h and weights θ, ϕ such that

$$\sum_{t=0}^{\infty} \lambda^t [\theta(x_t - \bar{x})^2 + \phi(\pi_t - \bar{\pi})^2] \equiv \sum_{t=0}^{\infty} \lambda^t [(x_t - x^*)^2 + \pi_t^2 + h[\beta(\pi_{t+1} - \bar{\pi}) - (\pi_t - \bar{\pi}) + f(x_t) - f(\bar{x})]] \quad (11)$$

up to a second order approximation in deviations about the steady state, give or take constant terms. Then the problem becomes that of minimizing (11) subject to

$$\pi_t - \bar{\pi} = \beta E_t \pi_{t+1} - \bar{\pi} - f(x_t) + f(\bar{x}) + u_t \cong \beta E_t \pi_{t+1} - \bar{\pi} + a(x_t - \bar{x}) + u_t \quad (12)$$

The BW procedure then amounts to finding the values $\bar{\pi}, \bar{x}, \theta, \phi$ and h which are consistent with the equalities in (11) and (12). For this simple example it is obvious that $\phi = 1$.

Using the Hamiltonian approach it is more straightforward to show that $\bar{\pi}, \bar{x}, h$ and θ are given by

$$2(\bar{x} - x^*) - hf'(\bar{x}) = 0 \quad 2\bar{\pi} + h \left(1 - \frac{\beta}{\lambda}\right) = 0 \quad (1 - \beta)\bar{\pi} - f(\bar{x}) = 0 \quad \theta = 1 + hb \quad (13)$$

Details are provided in Appendix A. By either method we can now see exactly what is wrong with minimizing (8) subject to (10). Suppose that the policymaker adopts the same discount factor as the private sector, then $\lambda = \beta$ and $\bar{\pi} = 0$; that is the steady state is the same deterministic non-inflationary steady state x , where $f(x) = 0$, we chose for (10). Then comparing the BW procedure with the standard LQ approximation discussed at the beginning of this section, we see that the latter is only a good approximation if $(x^* - \bar{x})$ or b are small. In the former case this implies that the output target is close to the non-inflationary stated state of x_t , whilst in the latter case the Phillips curve is nearly linear. If neither of these conditions apply then the BW or Hamiltonian procedures must be used.

3 The Social Planner's and Ramsey Problems

In this section we introduce the class of DSGE models to be studied. We assume a set of consumers, each with given endowments, whose objective is to maximize an intertemporal utility function. Typically this will incorporate consumption and leisure, but we shall state the objectives in a general fashion, so that they can incorporate habit as well. Thus the objective is for individual i to maximize an expected utility function of the form

$$E_0 \sum_{t=0}^{\infty} \beta^t u(W_{t-1}, W_{it}) \quad (14)$$

where the vector W_{it} represents individual i 's choices e.g. consumption and labour supply. This utility function may also incorporate habit or catching-up, and may therefore also be dependent on aggregate or average choices made in the previous period W_{t-1} .

We also assume that any resource constraints sum to a set of aggregate resource constraints. One can then define the social planner's problem in terms of the representative individual as that of

$$\max \sum_{t=0}^{\infty} \beta^t U(X_{t-1}, W_t) \quad \text{s.t.} \quad X_t = f(X_{t-1}, W_t) \quad (15)$$

where the set of constraints in this problem represent the set of (possibly intertemporal) resource constraints and exogenous processes describing the environment. These might include a dynamic equation for capital accumulation, and also capital utilization as in Smets and Wouters (2003).⁴

3.1 Characterization of the Efficient Level

Defining the Lagrangian

$$\sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \lambda_t^T (X_t - f(X_{t-1}, W_t))] \quad (16)$$

the following first order conditions provide the necessary conditions for the solution:

$$U_W(X_{t-1}, W_t) + \lambda_t^T f_W(X_{t-1}, W_t) = 0 \quad U_X(X_{t-1}, W_t) + \lambda_t^T f_X(X_{t-1}, W_t) - \frac{1}{\beta} \lambda_{t-1}^T = 0 \quad (17)$$

The steady state of the social planner's problem, the efficient level (denoted by $*$), is then given by

$$\begin{aligned} X^* &= f(X^*, W^*) & U_W(X^*, W^*) + \lambda^{*T} f_W(X^*, W^*) &= 0 \\ U_X(X^*, W^*) + \lambda^{*T} f_X(X^*, W^*) - \frac{1}{\beta} \lambda^{*T} &= 0 \end{aligned} \quad (18)$$

3.2 The Flexible-Price Solution and the Ramsey Problem

The difference between the efficient solution and that of the competitive or flexible-price solution is due to the externalities of habit and of firm and labour market power. As we shall see below for a particular example, the externality due to consumption habit works in the opposite direction to the externalities that produce the mark-ups in prices and wages.

⁴Although there appear to be significant differences in the functions u of (14) and U of (15), these are merely cosmetic. If we incorporate W_t as a subset of the state X_t , i.e. $X_t = [X_{1t}^T \ X_{2t}^T]$, where X_{1t} represents the resource constraints, and $X_{2t} = W_t$, then the presence of X_{t-1} in U is merely a generalization of including W_{t-1} in U .

In principle it is possible to set a proportional tax (or subsidy) in the flexible-price case that yields a ‘natural’ level of output exactly equal to the efficient level of output of the social planner.

Thus typically in economic models of this type we would assume monopolistic competition by firms. This leads to mark-up pricing, and creates a wedge between the level of output under competition - the natural rate - and the level of output that could be achieved by a social planner - the efficient level. This wedge may be exacerbated if we assume that there is labour market power as well. The latter is not incorporated by Benigno and Woodford (2005), but is common in most other New Keynesian models e.g. Clarida *et al.* (2002). We also assume that costs for firms are continuous, which rules out state-dependent $S - s$ policies; we do this because such policies cannot be easily aggregated. Initially we ignore the stochastic problem because the deterministic problem is sufficient to set up the LQ approximation.

Thus far we have only discussed the efficient and flexible-price levels of output. A more general model takes into account the fact that neither wages nor prices are completely flexible. As a consequence, we must discuss the case where a policymaker is required to maximize average welfare, in this case by choosing the optimal path for inflation. This is a particular case of the Ramsey problem.

Without going into details for the moment, the Ramsey problem differs from the social planner’s problem in that there is decentralized decision-making that has to be taken into account by the Ramsey planner. One can incorporate this in a general way into the model by expanding the state space to take account of the additional dynamic behaviour of the system. From the point of view of the Ramsey policymaker, the problem must be rewritten as one of maximizing

$$\sum_{t=0}^{\infty} \beta^t V(X_{t-1}, Z_{t-1}, W_t) \quad (19)$$

In the New Keynesian model that we study below, the only difference between the functions U and V is that there is an effect of price dispersion, where the latter is one of the components of the new set of variables Z_t . Price dispersion affects the disutility of labour. We assume that the decentralized decisions can somehow be aggregated, so that the constraints that must be satisfied by the Ramsey policymaker constitute both the resource constraints and the additional implementation constraints⁵ (typically associated with price and wage-setting, but excluding the intertemporal wealth constraint):

$$X_t = f(X_{t-1}, W_t) \quad Z_t = g(Z_{t-1}, X_{t-1}, W_t; \tau) \quad (20)$$

⁵This terminology is now widely used e.g. Khan *et al.* (2003), Kim *et al.* (2006b).

It is important to appreciate that the implementation constraints associated with Z_t represent individuals' and firms' decisions, and may involve future expectations. We take the approach that the Ramsey policymaker has a reputation for precommitment, so that we can take expectations of the future as always being fulfilled, and therefore regard all equations as backward looking. Suppose in addition that all factor prices are fixed so that inflation is 0 i.e. the appropriate elements of the vector Z are set equal to 0; we then obtain a solution to the 'natural' rate by solving for the steady state $\bar{X} = f(\bar{X}, \bar{W})$, $\bar{Z} = g(\bar{Z}, \bar{X}, \bar{W}; \tau)$. This is also known as the flexible price equilibrium. An important consideration is that the natural rate will be dependent on the tax/subsidy rate τ .

3.3 LQ Approximation of the Ramsey Problem: Efficient Case

Woodford (2003) now points out a key result for LQ-approximation. If at all possible, the aim of the Ramsey policymaker is to stabilize the economy about the efficient level of output. Let us assume therefore that the proportional tax/subsidy is set at exactly the level at which the flexible price equilibrium achieves the efficient level of output. This implies that there exists a value τ^* such that the efficient rate, coupled with zero inflation, is a solution to $Z^* = g(X^*, Z^*, W^*; \tau^*)$.

The main result of this section is dependent on the ability (a) to expand the utility function about the steady steady state efficient solution without the presence of linear terms and (b) to expand the constraints about the steady state efficient solution without the presence of constant terms.

Theorem 4:

The stabilization problem for the Ramsey policymaker can be approximately expressed as a quadratic expansion of the welfare function about the efficient level, provided that the Taylor series of $\tilde{V} = V(X_{t-1}, Z_{t-1}, W_t) - U(X_{t-1}, W_t)$ about the efficient level has no first-order terms.

Proof: We first deal with the utility function, using the notation for deviations from

steady state introduced earlier:

$$\begin{aligned}
& \sum_{t=0}^{\infty} \beta^t V(X_{t-1}, Z_{t-1}, W_t) = \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) + \tilde{V}(X_{t-1}, Z_{t-1}, W_t)] \\
& = \sum_{t=0}^{\infty} \beta^t [U(X_{t-1}, W_t) - \lambda^{*T}(X_t - f(X_{t-1}, W_t)) + \tilde{V}(X_{t-1}, Z_{t-1}, W_t)] \\
& \cong \sum_{t=0}^{\infty} \beta^t [U(X^*, W^*) + U_X x_{t-1} + U_W w_t - \lambda^{*T}(x_t - f_X x_{t-1} - f_W w_t) \\
& \quad + \frac{1}{2}(x_{t-1}^T H_{XX} x_{t-1} + 2x_{t-1}^T H_{XW} w_t + w_t^T H_{WW} w_t + \tilde{V}(X_{t-1}, Z_{t-1}, W_t)] \\
& = \sum_{t=0}^{\infty} \beta^t [U(X^*, W^*) + (U_X - \frac{1}{\beta} \lambda^{*T} + \lambda^{*T} f_X) x_{t-1} + (U_W + \lambda^{*T} f_W) w_t \\
& \quad + \frac{1}{2}(x_{t-1}^T H_{XX} x_{t-1} + 2x_{t-1}^T H_{XW} w_t + w_t^T H_{WW} w_t + \tilde{V}(X_{t-1}, Z_{t-1}, W_t)] \quad (21)
\end{aligned}$$

where $H = U(X, W) + \lambda^{*T} f(X, W)$, and its second derivatives are evaluated at (X^*, W^*) . Hence, using (18), the linear terms in x_t, w_t vanish. Since by assumption there are no first-order terms in the expansion of \tilde{V} at the efficient level, it follows that this expansion has no first-order terms present.

Now consider the constraints. Firstly the resource constraint is in steady state at the efficient level, so that an expansion about the latter will contain no constant term. Secondly, the constraint involving Z , by appropriate choice of $\tau = \tau^*$ is also in a zero-inflation steady state at the efficient level, so that any approximation of its dynamics about the efficient level will omit a constant term. This completes the proof.

The implication of this proof is that the welfare function cannot always be approximated as a constant plus quadratic terms, centred on the efficient rate, once the resource constraints have been incorporated. There are two conditions that must be satisfied for this approximation to be valid. Firstly the condition on \tilde{V} above needs to be checked; secondly the implementation constraints must incorporate a tax/subsidy rate such that their steady-state solution is characterized by zero inflation and the efficient level of output.

If the tax/subsidy rate is inconsistent with the above, then there is a distortion relative to the efficient case, and it is then necessary to assess whether this is a large or a small distortion.

3.4 The Small Distortion Case

Suppose that the tax/subsidy is insufficient to eliminate the inefficiency, but that the latter is small. There are then two approaches to obtain an approximation to the LQ

approximation. The first is take deviations about the inefficient steady state. This will, as we have seen above, produce an approximation to the welfare that contains a constant term (the steady-state welfare in the efficient case), and a quadratic term. The error in the approximation is then in the dynamic equation describing individual decisions. In this case, the efficient level Z^* is not consistent with the steady state because $\tau \neq \tau^*$, which means that the linearized approximation of the dynamic equation in for Z_t will contain a term $Z^* - g(X^*, Z^*, W^*; \tau)$; if this is small, it may be ignored.

The alternative is to take deviations about the natural rate, as done by Woodford (2003), Appendix E. Define the non-zero inflation natural rate as $(\bar{X}, \bar{Z}, \bar{W})$, which will be dependent on τ . The dynamic equations in deviation form then no longer contain a constant, but the linear terms in the welfare approximation (21) are now of the form:

$$\begin{aligned}
& (U_X(\bar{X}, \bar{W}) - \frac{1}{\beta}\lambda^{*T} + \lambda^{*T} f_X(\bar{X}, \bar{W}))x_{t-1} + (U_W(\bar{X}, \bar{W}) + \lambda^{*T} f_W(\bar{X}, \bar{W}))w_t \\
& \cong (H_X + (\bar{X} - X^*)^T H_{XX} + (\bar{W} - W^*)^T H_{WX})x_{t-1} \\
& \quad + (H_W + (\bar{X} - X^*)^T H_{XW} + (\bar{W} - W^*)^T H_{WW})w_t \\
& = ((\bar{X} - X^*)^T H_{XX} + (\bar{W} - W^*)^T H_{WX})x_{t-1} \\
& \quad + ((\bar{X} - X^*)^T H_{XW} + (\bar{W} - W^*)^T H_{WW})w_t
\end{aligned} \tag{22}$$

Thus the linear terms can be ignored provided that $\bar{X} - X^*$ and $\bar{W} - W^*$ are small.

To summarize then, there are two ways of assessing whether 'small distortions' are indeed small, and which relate directly to the necessary conditions examined in Section 2. This is done either by (1) evaluating the effect on the constant in the aggregated decentralized equations or (2) the effect on the first-order terms in the Hamiltonian. A further method of assessing the limitations of the small distortion case is discussed in the next section, by comparing the weights on the quadratic terms of the LQ welfare approximation for the efficient and the non-efficient case. This provides an arguably more direct assessment of the error in the approximation; this is because it is less easy to assess the impact of the errors described above.

4 LQ Approximation of Optimal Monetary Policy in a DSGE Model

We now turn to the main model of the paper and to optimal monetary policy. We initially investigate the large distortions approximation and towards the end of this section, we study both the efficient and the small distortions case.

The standard New Keynesian model ascribes a fixed probability in each period of changing prices (and wages). This leads to dynamic equations for the overall price index,

and in turn this leads in the Woodford (2003) case to different choices of labour supply by individuals, and in the Clarida *et al.* (2002) case to each individual providing the same quantity of labour. In the former, the policymaker takes the average of the utility function, which for small variance of shocks is approximately the same as flexible-price level of the utility function, but with an additional effect from the spread of prices. In the latter, although labour supply is the same for each worker, it is dependent on the spread of demand for each good; this in turn leads to the utility function differing from the flexible price utility function by a term dependent on the spread of prices and wages.

The model is of a cashless economy with external habit in consumption. Consumers of type i maximize the intertemporal trade-off between consumption C_{it} - taking into account a desire to consume at a level similar to that of last period's average consumption C_{t-1} - and leisure. The latter is accounted for by penalising working time N_{it} .

Unlike Clarida *et al.* (2000) we do not incorporate a proportional tax (or subsidy) into the model in order to ensure that the steady state, or natural rate, of output is at the efficient level. Instead we use the methodology of Section 2 to obtain a quadratic approximation to the welfare when the natural rate differs from the efficient rate. This is an issue also addressed by Benigno and Woodford (2005) using the less direct methods outlined in the example of Section 2.

We can summarize the model in a concise form as:

Household Utility:

$$\Omega_0 = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left[\frac{(C_{it} - h_C C_{t-1})^{1-\sigma}}{1-\sigma} - \kappa \frac{N_{it}^{1+\phi}}{1+\phi} \right] \right] \quad (23)$$

Household Behaviour:

The first-order conditions for households are as follows:

$$1 = \beta E_t \left[D_{t,t+1} \left(\frac{C_{it+1} - h_C C_t}{C_{it} - h_C C_{t-1}} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right] \quad (24)$$

$$\frac{W_{it}}{P_t} = \frac{\kappa}{(1 - \frac{1}{\eta})} N_{it}^{\phi} (C_{it} - h_C C_{t-1})^{\sigma} \quad (25)$$

where $D_{t,t+1}$ is the stochastic discount factor on holdings of one-period bonds, and the gross inflation rate Π_t is given by

$$\Pi_t \equiv \frac{P_t}{P_{t-1}} \quad (26)$$

All consumers can trade in a complete set of state contingent bonds, and therefore engage in complete risk-sharing, so that (24) represents the Keynes-Ramsey intertemporal first-order condition for consumption across all consumers, taking habit into account. (25)

equates relative marginal utilities of consumption and leisure to the real wage. W_{it}, P_t are measures of the nominal wage of the i th agent and of price respectively. (25) also incorporates market power of individual consumers, who are all distinct from the point of view of production skills and face a demand curve

$$N_{it} = \left(\frac{W_{it}}{W_t} \right)^{-\eta} N_t \quad (27)$$

where W_t and N_t denote aggregates, $N_t \equiv \sum_i N_{it}^{\frac{\eta-1}{\eta}}$. Aggregate output Y_t is similarly defined by aggregating over all labour inputs.

There is market-clearing in wages, and in this set-up all agents set the same wage and work the same number of hours. Thus (25) holds when i is deleted, so for this setup there is no need to aggregate W_t, N_t .

Firms:

Unlike workers, firms only reset prices in any given period with probability $1 - \xi$. Thus the optimal price P_t^0 for any firm that sets its price at t must take into account any future periods during which the price remains unchanged.⁶

The first-order condition for profit-maximization for the j th firm over the duration of the optimal price not being reset takes into account the elasticity of substitution ζ between goods, which provides firms with monopolistic power. It is given by

$$P_t^0 E_t \left[\sum_{k=0}^{\infty} \xi^k D_{t,t+k} Y_{t+k}(j) \right] = \frac{\kappa}{(1 - 1/\zeta)} E_t \left[\sum_{k=0}^{\infty} \xi^k D_{t,t+k} P_{t+k} MC_{t+k} Y_{t+k}(j) \right] \quad (28)$$

where marginal cost is given by the real product wage $MC_t = \frac{W_t}{A_t P_t}$ and the stochastic discount factor $D_{t,t+k}$ is given by

$$D_{t,t+k} = \beta^k \left(\frac{C_{t+k} - h_C C_{t+k-1}}{C_t - h_C C_{t-1}} \right)^{-\sigma} \frac{P_t}{P_{t+k}} \quad (29)$$

The first-order condition (28) is cumbersome to manipulate. However it is possible to express this price-setting rule in terms of *difference equations* that are far easier to manipulate. To do this first note that

$$Y_{t+k}(j) = \left(\frac{P_t^0}{P_{t+k}} \right)^{-\zeta} Y_{t+k} \quad (30)$$

⁶It is easy to show that if there is planned indexation to the overall price index as well i.e. the future price at time $t+k$ is given by $P_t^0 (P_{t+k-1}/P_t)^\gamma$ then all the results presented here are the same when Π_t is replaced by Π_t/Π_{t-1}^γ .

and multiplying both sides of (28) by $(\frac{P_t^0}{P_t})^\zeta (C_t - h_C C_{t-1})^{-\sigma}$ and in addition noting that $P_{t+k}/P_t = \Pi_{t+k} \dots \Pi_{t+1}$, the firms' staggered price setting can be succinctly described by

$$Q_t = \Lambda_t / H_t \quad (31)$$

where we have defined variables Q_t , H_t and Λ_t by

$$Q_t \equiv P_t^0 / P_t \quad (32)$$

$$H_t - \xi \beta E_t[\Pi_{t+1}^{\zeta-1} H_{t+1}] = Y_t (C_t - h_C C_{t-1})^{-\sigma} \quad (33)$$

$$\Lambda_t - \xi \beta E_t[\Pi_{t+1}^\zeta \Lambda_{t+1}] = \frac{\kappa}{(1 - 1/\zeta)(1 - 1/\eta)} Y_t N_t^\phi \quad (34)$$

Price index inflation given by

$$1 = \xi \Pi_t^{\zeta-1} + (1 - \xi) Q_t^{1-\zeta} \quad (35)$$

The production function at the firm level is defined as

$$Y_t(j) = A_t N_t(j) \quad (36)$$

where A_t represents a common technology shock and $N_t(j)$ is an aggregate of differentiated labour chosen by firm j . Then aggregate output across firms, some of whom can re-optimize prices at time t , is given by $Y_t = A_t \sum_j N_t(j) = A_t N_t$.

4.1 Price Dispersion and the Costs of Inflation

Here we discuss the effects of inflation on the dispersion of prices due to firms' behaviour discussed above, and the implications for total employment. These dispersion effects will lead to costs of inflation, as we shall see later.

The impact of price dispersion arises from labour input being the same for each individual, but dependent on demand for each good:

$$N_t = \sum_j N_t(j) = \frac{Y_t}{A_t} \sum_j \frac{Y_t(j)}{Y_t} = \frac{Y_t}{A_t} \sum_j \left(\frac{P_t(j)}{P_t} \right)^{-\zeta} \quad (37)$$

Now define the effect of price dispersion on labour demand as $D_t = \sum_j (\frac{P_t(j)}{P_t})^{-\zeta}$. Assuming that the number of firms is large, we can use the law of large numbers to obtain the following dynamic relationship:

$$D_t = \xi \Pi_t^\zeta D_{t-1} + (1 - \xi) Q_t^{-\zeta} \quad (38)$$

4.2 The Ramsey Problem

For simplicity, we ignore government spending, so that $Y_t = C_t$. As a consequence of the price diversion result above, the problem for a policy maker is characterized by solving the *deterministic* ex ante commitment (and Ramsey) problem by choosing a trajectory for inflation to maximize

$$\Omega_0 = \sum_{t=0}^{\infty} \beta^t \left[\frac{(Y_t - Z_t)^{1-\sigma}}{1-\sigma} - \frac{\kappa}{1+\phi} \left(\frac{Y_t}{A_t} \right)^{1+\phi} D_t^{1+\phi} \right] \quad (39)$$

subject to the constraints (38) and

$$Z_t = h_C Y_{t-1} \quad 1 = \xi \Pi_t^{\zeta-1} + (1-\xi) Q_t^{1-\zeta} \quad Q_t H_t = \Lambda_t \quad (40)$$

$$H_t - \xi \beta E_t[\Pi_{t+1}^{\zeta-1} H_{t+1}] = Y_t (Y_t - Z_t)^{-\sigma} \quad (41)$$

$$\Lambda_t - \xi \beta E_t[\Pi_{t+1}^{\zeta} \Lambda_{t+1}] = \frac{\kappa}{(1-1/\zeta)(1-1/\eta)} \left(\frac{Y_t}{A_t} \right)^{1+\phi} D_t^{\phi} \quad (42)$$

Note that the dynamics of the term D_t^{ϕ} contains only second-order terms, and therefore satisfies Theorem 2. Then had we included an optimal subsidy rate, we would have been able to expand the utility function about the efficient rate.

We can now write the Lagrangian for the policymaker's optimal control problem as follows:

$$\begin{aligned} L = & \Omega_0 + \sum_{t=0}^{\infty} \beta^t [\lambda_{1t}(Z_{t+1} - h_C Y_t) + \lambda_{2t}(1 - \xi \Pi_t^{\zeta-1} - (1-\xi) Q_t^{1-\zeta}) \\ & + \lambda_{3t}(Q_t H_t - \Lambda_t) + \lambda_{4t}(H_t - \xi \beta \Pi_{t+1}^{\zeta-1} H_{t+1} - Y_t (Y_t - Z_t)^{-\sigma}) \\ & + \lambda_{5t}(\Lambda_t - \xi \beta \Pi_{t+1}^{\zeta} \Lambda_{t+1} - \frac{\kappa}{\alpha} \left(\frac{Y_t}{A_t} \right)^{1+\phi} D_t^{\phi}) \\ & + \lambda_{6t}(D_t - \xi \Pi_t^{\zeta} D_{t-1} - (1-\xi) Q_t^{-\zeta})] \end{aligned} \quad (43)$$

where we define $\alpha = (1-1/\zeta)(1-1/\eta)$.

First-order conditions are given by:

$$\begin{aligned} & (Y_t - Z_t)^{-\sigma} - \kappa \frac{Y_t^{\phi}}{A_t^{1+\phi}} D_t^{1+\phi} - \lambda_{1t} h_C \\ & - \lambda_{5t} \frac{\kappa(1+\phi)}{\alpha} \frac{Y_t^{\phi}}{A_t^{1+\phi}} D_t^{\phi} - \lambda_{4t} ((Y_t - Z_t)^{-\sigma} - \sigma Y_t (Y_t - Z_t)^{-\sigma-1}) = 0 \\ & -(Y_t - Z_t)^{-\sigma} + \frac{1}{\beta} \lambda_{1t-1} - \lambda_{4t} \sigma Y_t (Y_t - Z_t)^{-\sigma-1} = 0 \\ & \beta(1-\zeta) \xi \lambda_{2,t+1} \Pi_{t+1}^{\zeta-2} - \lambda_{4t} \xi \beta (\zeta-1) \Pi_{t+1}^{\zeta-2} H_{t+1} - \lambda_{5t} \xi \beta \zeta \Pi_{t+1}^{\zeta-1} \Lambda_{t+1} - \xi \beta \zeta \lambda_{6,t+1} \Pi_{t+1}^{\zeta-1} D_t = 0 \\ & - \lambda_{2t} (1-\xi) (1-\zeta) Q_t^{-\zeta} + \lambda_{3t} H_t + \zeta (1-\xi) \lambda_{6t} Q_t^{-\zeta-1} = 0 \end{aligned}$$

$$\begin{aligned}
\lambda_{3t}Q_t + \lambda_{4t} - \xi\Pi^{\zeta-1}\lambda_{4,t-1} &= 0 \\
-\lambda_{3t} + \lambda_{5t} - \xi\Pi_t^\zeta\lambda_{5,t-1} &= 0 \\
-\kappa\left(\frac{Y_t}{A_t}\right)^{1+\phi}D_t^\phi + \lambda_{6t} - \xi\beta\Pi_{t+1}^\zeta\lambda_{6,t+1} - \frac{\kappa\phi}{\alpha}\left(\frac{Y_t}{A_t}\right)^{1+\phi}D_t^{\phi-1}\lambda_{5t} &= 0
\end{aligned}$$

The zero-inflation equilibrium values are given by

$$\Pi = Q = 1 \quad \Lambda = H = \frac{Y^{1-\sigma}(1-h_C)^{-\sigma}}{1-\beta\xi} \quad D = 1 \quad (1-h_C)^{-\sigma} = \frac{\kappa}{\alpha} \frac{Y^{\phi+\sigma}}{A^{1+\phi}} \quad (44)$$

$$\lambda_5 = \frac{1-\beta h_C - \alpha}{\frac{\sigma(1-h_C\beta)}{1-h_C} + \phi} = -\lambda_4 \quad \lambda_3 = (1-\xi)\lambda_5 \quad \lambda_2 = \frac{H\lambda_5 + \zeta\lambda_6}{(1-\zeta)} \quad (45)$$

$$\lambda_6 = \frac{Y^{1-\sigma}(1-h_C)^{-\sigma} \left(\frac{\alpha\sigma}{1-h_C} + \phi\right)(1-h_C\beta)}{1-\beta\xi \frac{\sigma(1-h_C\beta)}{1-h_C} + \phi} \quad (46)$$

Now that we have the steady-state values of the Lagrange multipliers, we are in a position to apply Theorem 1. We first linearize the relationships between the variables, and then obtain the quadratic approximation of the Lagrangian. We shall leave discussion of Theorem 3 till later.

4.3 Linearization of Dynamics

We linearize about a zero-inflation steady state and later examine the second-order conditions for this to be appropriate for our LQ approximation. Define $h_t, \lambda_t, q_t, \pi_t$ as deviations of $H_t, \Lambda_t, Q_t, \Pi_t$ from their steady state values. In addition define $y_t = (Y_t - Y)/Y \approx \log Y_t/Y$, $a_t = (A_t - A)/A$ and define $z_t = (Z_t - Z)/Y$.

Linearization of the constraints yields

$$Hq_t = \lambda_t - h_t \quad \xi\pi_t = (1-\xi)q_t \quad (47)$$

$$z_{t+1} = h_C y_t \quad (48)$$

$$h_t - \beta\xi(\zeta-1)HE_t\pi_{t+1} - \beta\xi E_t h_{t+1} = Y^{1-\sigma}(1-h_C)^{-\sigma} \left(y_t - \frac{\sigma}{1-h_C}(y_t - z_t)\right) \quad (49)$$

$$\lambda_t - \beta\xi\zeta\Lambda E_t\pi_{t+1} - \beta\xi E_t \lambda_{t+1} = \frac{\kappa(1+\phi)}{\alpha} \frac{Y^{1+\phi}}{A^{1+\phi}} (y_t - a_t) \quad (50)$$

Now subtract (49) from (50). Noting that $\Lambda = H$, and substituting from (47) yields a Phillips curve relationship of the form:

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} \left(\phi y_t + \frac{\sigma}{1-h_C}(y_t - z_t) - (1+\phi)a_t\right) \quad (51)$$

Note that linearization of the dispersion term around zero inflation is irrelevant, since it reduces to $d_t - \xi d_{t-1} = 0$.⁷ Also note that a_t can be a stochastic process turning the optimization problem into one that is stochastic.

4.4 The Accurate LQ Approximation

At this point we apply the result of Section 2, in order to obtain a quadratic approximation to the period t value of the Lagrangian. Ignoring the steady state value of the latter, the remaining terms are given by:

$$\begin{aligned}
& -\frac{1}{2}(Y - Z)^{-\sigma-1}Y^2\sigma(y_t - z_t)^2 - \frac{1}{2}\kappa\phi\frac{Y^{1+\phi}}{A^{1+\phi}}y_t^2 - \lambda_5\frac{\kappa}{2\alpha}\phi(1 + \phi)\frac{Y^{1+\phi}}{A^{1+\phi}}y_t^2 \\
& + \kappa(1 + \phi)\frac{Y^{1+\phi}}{A^{1+\phi}}y_t a_t + \lambda_5\frac{\kappa}{\alpha}(1 + \phi)^2\frac{Y^{1+\phi}}{A^{1+\phi}}y_t a_t \\
& - \lambda_5\sigma Y^2(Y - Z)^{-\sigma-1}(y_t - z_t)y_t + \frac{1}{2}\lambda_5\sigma(\sigma + 1)Y^3(Y - Z)^{-\sigma-2}(y_t - z_t)^2 \\
& - \frac{\xi}{2}\pi_t^2((\zeta - 1)(\zeta - 2)\lambda_2\Pi^{\zeta-3} + (\zeta - 1)(\zeta - 2)\Pi^{\zeta-3}H\lambda_4 + \zeta\lambda_5(\zeta - 1)\Pi^{\zeta-2}\Lambda \\
& + \lambda_6\zeta(\zeta - 1)\Pi^{\zeta-2}D) - \xi\pi_t\lambda_t\zeta\lambda_5\Pi^{\zeta-1} - \xi\pi_t h_t(\zeta - 1)\lambda_4\Pi^{\zeta-2} \\
& + \frac{1}{2}q_t^2(\lambda_2(1 - \xi)(1 - \zeta)\zeta Q^{-1-\zeta} + \lambda_6(1 - \xi)(1 + \zeta)\zeta Q^{-2-\zeta}) + q_t h_t \lambda_3 \tag{52}
\end{aligned}$$

After eliminating h_t, λ_t, q_t using (47), and substituting the steady state values above, we finally arrive at the *correct quadratic approximation to the non-linear Ramsey problem* as the maximization of $E_0 [\sum_{t=0}^{\infty} \beta^t U_t]$ with respect to $\{\pi_t\}$, subject to (51) where

$$\begin{aligned}
U_t = & -\frac{\kappa}{2\alpha}\frac{Y^{1+\phi}}{A^{1+\phi}}\left[\frac{\sigma}{1 - h_C}(y_t - h_C y_{t-1})^2 + \phi(\alpha + \lambda_5(1 + \phi))y_t^2 \right. \\
& - 2(1 + \phi)(\alpha + \lambda_5(1 + \phi))y_t a_t + 2\lambda_5\frac{\sigma}{1 - h_C}(y_t - h_C y_{t-1})y_t \\
& \left. - \lambda_5\frac{\sigma(\sigma + 1)}{(1 - h_C)^2}(y_t - h_C y_{t-1})^2 + \frac{\xi\zeta}{(1 - \xi)(1 - \beta\xi)}(\alpha + (1 + \phi)\lambda_5)\pi_t^2\right] \tag{53}
\end{aligned}$$

4.5 The Social Planner's Problem

The Social Planner can be regarded as maximizing (23) viewing all agents as identical, and so can set $C_{it} = C_t, N_{it} = N_t$, subject to the constraint $C_t = Y_t = A_t N_t$. The social planner chooses a trajectory for output which satisfies the first-order condition

$$[C_t - h_C C_{t-1}]^{-\sigma} - h_C \beta [C_{t+1} - h_C C_t]^{-\sigma} = \kappa \frac{Y_t^\phi}{A_t^{1+\phi}} \tag{54}$$

⁷As Kim *et al.* (2006a) show, this feature follows from the particular choice of variables with respect to which we applied the Taylor series approximation. If we had chosen a different normalization and linearized with respect to $\sqrt{\log D_t/D}$ instead, we would have a bifurcation problem.

The efficient steady-state level of output $Y_{t+1} = Y_t = Y_{t-1} = Y^*$, say, is therefore given by

$$(Y^*)^{\phi+\sigma} = \frac{(1 - h_C\beta)A^{1+\phi}}{\kappa(1 - h_C)^\sigma} \quad (55)$$

We can now examine the inefficiency of the zero-inflation steady state. From (44) the zero-inflation steady state output in the Ramsey problem is given by $Y = \bar{Y}$ where

$$(\bar{Y})^{\phi+\sigma} = \frac{\left(1 - \frac{1}{\zeta}\right) \left(1 - \frac{1}{\eta}\right) A^{1+\phi}}{\kappa(1 - h_C)^\sigma} \quad (56)$$

It is easy to check that this is exactly the same steady-state level as that of the flexi-price economy where firms set prices optimally at every period. Comparing (55) and (56) we have the result first obtained by Choudhary and Levine (2006):

Result 1

The natural level of output, \bar{Y} , is below the efficient level, Y^* , if and only if

$$\alpha \equiv \left(1 - \frac{1}{\zeta}\right) \left(1 - \frac{1}{\eta}\right) < 1 - h_C\beta \quad (57)$$

In the case where there is no habit persistence in consumption, $h_C = 0$, then (57) always holds. In this case market power in the output and labour markets captured by the elasticities η, ζ respectively drive the natural rate of output below the efficient level. If habit persistence in consumption is sufficiently high, then (57) does not hold and the natural rate of output and employment are then too high compared with the efficient outcome and people are working too much. Why is this? In the efficient case, there is an incentive for the social planner to raise C_t relative to $h_C C_{t-1}$, but also a disincentive to raise C_t because of its effect on welfare in the next period. For decentralized consumers, there is no disincentive effect because each will ignore the effect of its current raised consumption in the next period. The greater is h_C the greater is the effect of the disincentive on the social planner.

Is there empirical support that (57) holds? Terms $\left(1 - \frac{1}{\zeta}\right)$ and $\left(1 - \frac{1}{\eta}\right)$ are the inverses of mark-ups over marginal costs in the output and labour markets respectively. A plausible upper bound on these mark-ups is 20% so $\alpha = \left(1 - \frac{1}{\zeta}\right) \left(1 - \frac{1}{\eta}\right) > \frac{1}{1.2^2}$. A condition on h_C for (57) to hold is therefore $h_C\beta < 0.306$. Most empirical estimates of habit in a quarterly model are in the range $h_C = [0.5, 0.9]$ which would see this condition not holding.⁸

⁸If we were to add a tax wedge T , then (57) becomes $(1 - T) \left(1 - \frac{1}{\zeta}\right) \left(1 - \frac{1}{\eta}\right) < 1 - h_C\beta$ which suggests that the condition would hold for a tax wedge $T > 0.5$ as observed in the euro-area. See Levine *et al.* (2006).

4.6 The Efficient Case

For this particular example, we are able to illustrate Theorem 4 without needing directly to invoke a tax/subsidy and add a further algebraic burden. Suppose instead that the effect of habit is to directly offset the effect of the distortions due to η, ζ i.e. $1 - \alpha - \beta h_C = 0$, so that the value of output in the Ramsey problem (assuming zero inflation) is equal to the efficient level. We need to check that the sufficient condition on \tilde{V} of Theorem 4 holds in this case. One can see by inspection, that to first order about the efficient level of output and zero inflation, we have the following expansion:

$$\sum_{t=0}^{\infty} \beta^t \tilde{V}_t = \sum_{t=0}^{\infty} -\beta^t \frac{\kappa}{1+\phi} \left(\frac{Y_t}{A_t} \right)^{1+\phi} (D_t^{1+\phi} - 1) \cong \sum_{t=0}^{\infty} -\beta^t \kappa \left(\frac{Y}{A} \right)^{1+\phi} d_t \quad (58)$$

But to first order, it is easy to see that $d_t = \xi d_{t-1}$, so starting at $d_0 = 0$, it is clear that to first order, the value of (58) is zero.

It is now of interest to verify that expansion to second order in this case yields the same expansion as for the large distortions case. We first note that the second order expansion of D_t is given by

$$d_t = \xi d_{t-1} + \frac{1}{2} \frac{\xi \zeta}{1 - \xi} \pi_t^2 \quad (59)$$

from which it follows that in a second order expansion we can ignore d_t^2 . Furthermore, using (58) and (59), it follows that to second order about the efficient solution, we have

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \left[\frac{(Y_t - Z_t)^{1-\sigma}}{1-\sigma} - \frac{\kappa}{1+\phi} \left(\frac{Y_t}{A_t} \right)^{1+\phi} D_t^{1+\phi} \right] \cong \\ -\frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left[\frac{\sigma Y^{*1-\sigma}}{(1-h_C)^{1+\sigma}} (y_t - z_t)^2 + \kappa \phi \left(\frac{Y^*}{A} \right)^{1+\phi} y_t^2 + \frac{\kappa \xi \zeta}{(1-\beta\xi)(1-\xi)} \left(\frac{Y^*}{A} \right)^{1+\phi} \pi_t^2 \right] \end{aligned} \quad (60)$$

We now need to compare this with the expansion in the large distortions case, given by (53). Using (55) in (60), and using the fact that $\lambda_5 = 0$ (because $1 - \alpha - \beta h_C = 0$) in (53) for this case, it is easy to show that the quadratic expansions are identical.

4.7 The Small Distortion Case

The small distortion case assumes that the zero-inflation steady state about which we have linearized is approximately efficient. From result 2 this implies that $1 - \beta h_C - \alpha$ is small. We are now in a position to examine the nature of this ‘approximation to an approximation’ by examining the correctly quadratified single-period utility (53). From (45) we can see that this means that λ_5 must be small. An examination of (53) reveals

that the small distortion case, which would omit all terms involving λ_5 , is valid only if $|\lambda_5(1 + \phi)| \ll \alpha$ or, using the definition of λ_5 , only if

$$(1 + \phi) \frac{|1 - \beta h_C - \alpha|}{\frac{\sigma(1 - h_C \beta)}{1 - h_C} + \phi} \ll \alpha \quad (61)$$

Typical estimated parameter values are $\sigma = 3$ (with this value or higher being confirmed within other contexts as well), $\phi = 1.3$. With h_C at the mid-point of the range of estimates at $h_C = 0.7$ this gives the left-hand-side of (61) as 0.22 and the right-hand side as 0.69. Neglected terms are therefore of the order of one third of those retained.

5 Target Implementability and External Habit

Svensson (2003, 2005) suggests real-world monetary policy is best viewed in terms of a “prescribed guide for monetary policy”. These would include “targeting rules” and “instrument rules”. The latter could consist of Taylor-type that prescribe the commitment of the monetary authority to change the nominal interest rate in response to changes in target macro-economic variables such as the output gap and past, current or expected future inflation rates. However on both normative and positive grounds he strongly argues for the former.⁹ A general targeting rules would specify the objectives to be achieved by for example setting out target variables, their targets and a loss function to be minimized. In the context of our quadratic approximations we can interpret these targets as ‘bliss points’, provided that the period t quadratic approximation achieves a maximum at these. This leads to a particular form of targeting rule in the Svensson sense that we call ‘target-implementability’:

Definition: A period- t welfare function is **target-implementable** iff, in the vicinity of the steady state of the first-order conditions for a maximum, it can be written as a weighted sum of squares of linear terms, with all weights negative; that is a sum of the squares of deviations of target variables from their bliss points.

From Theorem 2 and the fact that a symmetric positive definite matrix Q can be written as $Q = X\Lambda X'$ where X is a matrix of eigenvectors and Λ is a diagonal matrix of real positive eigenvalues we now have:

⁹This paper does not engage with the targeting versus instrument rules debate (but see, for example, McCallum and Nelson (2004)).

Result 2

A necessary and sufficient condition for the solution to the Ramsey problem to be target-implementable is that, in the vicinity of the steady state of the first-order conditions for a maximum, the quadratic approximation to the Lagrangian is negative semi-definite.

Now consider the target-implementability of the welfare function in our LQ approximation to the Ramsey problem. First consider the case without habit ($h_C = 0$). After some further effort (and subtracting an appropriate term in a_t^2), (53) then reduces to

$$\frac{-\kappa Y^{\phi+1}}{2\alpha A^{1+\phi}}(\phi + \sigma\alpha + 1 - \alpha) \left[\left(y_t - \frac{1+\phi}{\sigma+\phi} a_t \right)^2 + \frac{\zeta\xi}{(1-\xi)(1-\beta\xi)(\sigma+\phi)} \pi_t^2 \right] \quad (62)$$

This is clearly target-implementable with a stochastic output target $\frac{1+\phi}{\sigma+\phi} a_t$ and inflation target of zero (the steady state about which we have formed the LQ approximation). Since from Theorem 3 and result 2, the condition for target-implementability is a sufficient condition for the first-order conditions to define a local maximum, we can now also confirm that the zero inflation steady state (that we found to satisfy these first-order conditions for $h_C \geq 0$) is indeed appropriate. Our stochastic target is of course the flexi-price output so that (62) turns out to be micro-founded loss function popularized by Woodford (2003) that penalizes deviations of the output gap and inflation from zero.

Now consider the case $h_C > 0$. As for $h_C = 0$ we now need to demonstrate whether the natural rate as calculated, with zero inflation, is actually the steady state for the Ramsey problem. To check this, we need either to solve the corresponding Riccati equation or to check the sufficient conditions of target-implementability. If the sufficient conditions of the latter are not satisfied, then checking the steady state Riccati matrix will not yield analytic results. This is because the equation governing it is highly nonlinear, and in addition the matrix is of dimension 2, so analytic solutions will not in general be found.

We therefore focus on target-implementability, and determine what conditions on the underlying parameters are required for (53) to be negative semi-definite. By inspecting this approximation we can see that apart from completing the square for the disturbance term a_t , it will be negative semi-definite provided that the terms in y_t and $(y_t - z_t)$ are negative semi-definite. This is equivalent to the requirement that we can write these terms as a weighted sum of y_t^2 and $(y_t - \nu z_t)^2$, for some ν .

In order to reduce the algebraic burden for checking target-implementability, we make the relatively innocuous approximation $\beta = 1$, since most quarterly models would assume a value of the order of 0.99. It turns out that even then, the conditions for target-implementability are rather messy, so instead we focus on a set of empirically innocuous sufficient conditions on the parameters.

Proposition

(i) When (57) does not hold so that the natural rate is above the efficient rate, sufficient conditions for the Ramsey problem with habit in consumption to have a target-implementable zero inflation steady state are that $\sigma > 1$ and $\phi\sigma^2 > \phi + \sigma$.

(ii) When (57) holds so that the natural rate is below the efficient rate, sufficient conditions for the zero-inflation steady state to be target-implementable are that $\alpha(1 + \sigma) > (1 - h_C)(1 - \phi)$ and $\phi^3 > \phi + \sigma$.

Proof: See Appendix B.

We note that for either case (i) or (ii), the single-period welfare loss may be rewritten

$$\begin{aligned}
 & -\frac{\kappa}{2\alpha} \frac{Y^{1+\phi}}{A^{1+\phi}} \left[a_1(y_t - h_C y_{t-1} + \frac{a_2}{a_1} y_t)^2 + (a_3 - \frac{a_2^2}{a_1}) y_t^2 - 2(1 + \phi)(\alpha + \lambda_5(1 + \phi)) y_t a_t \right. \\
 & \left. + \frac{\xi\zeta}{(1 - \xi)(1 - \beta\xi)} (\alpha + (1 + \phi)\lambda_5) \pi_t^2 \right] \\
 a_1 = & \frac{\sigma}{1 - h_C} - \lambda_5 \frac{\sigma(\sigma + 1)}{(1 - h_C)^2} \quad a_2 = \lambda_5 \frac{\sigma}{1 - h_C} \quad a_3 = \phi(\alpha + \lambda_5(1 + \phi)) \quad (63)
 \end{aligned}$$

with $a_1 > 0$, $a_3 - \frac{a_2^2}{a_1} > 0$. Thus at each period there is a bliss-point for inflation of 0, a bliss-point for output of $(1 + \phi)(\alpha + \lambda_5(1 + \phi))/(a_3 - \frac{a_2^2}{a_1}) a_t$, and a bliss-point for output growth $y_t - y_{t-1}$ in terms of last period's output given by $-\frac{(1 + \frac{a_2}{a_1} - h_C)}{(1 + \frac{a_2}{a_1})} y_{t-1}$.¹⁰ In the special case $h_C = 0$, there is no bliss-point for output growth, the coefficient on y_t^2 is $a_1 + 2a_2 + a_3$, so that the bliss-point for output is $\frac{1+\phi}{\sigma+\phi} a_t$ as in (62).

Using typical estimated parameter values discussed above, both of the sufficient conditions (i) and (ii) are easily satisfied. Necessary conditions are much more difficult to derive, as there is a wide range of parameter values for which the sufficient conditions of Theorem 3 are satisfied, even though the quadratic approximation is not negative definite. The following set of theoretically possible parameters however, yields a situation where the necessary first-order conditions yield a zero-inflation steady state, but there is no steady-state for the Riccati matrix: $\sigma < 0.6$, $\phi = 0.03$, $h_C = 0.75$, $\xi = 0.7$ and $\zeta = 5$, $\eta = 16$ (implying $\alpha = 0.75$). Furthermore, the sufficient conditions of Theorem 3 are violated for some values of t . Thus we have the following:

Result 3: For the problem with habit, there are possible configurations of parameter values such that the solution to the Ramsey problem does not have a zero inflation steady-state equilibrium.

¹⁰Note that $a_2 \geq 0$ iff $\lambda_5 \geq 0$. $\lambda_5 = 0$ for the efficient case; $\lambda_5 > 0$ if h_C is small and $\lambda_5 < 0$ as $h_C \rightarrow 1$.

Although $\sigma < 0.6$ lies outside most, but not all estimates, this example is interesting because it serves as an example of why second-order conditions, routinely by-passed in the optimal dynamic policy literature, do matter. Benigno and Benigno (2006) discuss a case where randomized policy may be superior in a non-inflationary steady state. The conditions they derive for this correspond precisely to those which guarantee that the steady-state solution to the corresponding Riccati equation is either negative definite, or in which the negative definite solution is unstable. Only by implication do they suggest that the zero-inflation optimal solution must therefore be completely invalid.

6 Nash Equilibria in a Two-Bloc Model

We now investigate a two bloc version of our previous model without habit in consumption as in Clarida *et al.* (2002), and focus on the use of the subsidy approach which enables the decentralized level of output to match the efficient level. For this case we abstract from productivity shocks and choose units so that the production functions are $Y = N, Y^* = N^*$ for the home and foreign blocs respectively. Thus we can regard the level of output in each bloc as the resource.

The two blocs are of different sizes, in the ratio $(1 - \gamma) : \gamma$. Consumption preferences are Cobb-Douglas, and once we take the risk-sharing relationship into account, this results in terms-of-trade being equal to relative output.

6.1 Non-cooperative Social Planners' Problem

From the viewpoint of the social planner in each country, each has to solve the following problems:

The home planner maximizes

$$\frac{C^{1-\sigma}}{1-\sigma} - \frac{Y^{1+\phi}}{1+\phi} \quad s.t. \quad C = kY^{1-\gamma}Y^{*\gamma} \quad (64)$$

while the foreign planner maximizes

$$\frac{C^{*1-\sigma}}{1-\sigma} - \frac{Y^{*1+\phi}}{1+\phi} \quad s.t. \quad C^* = kY^{1-\gamma}Y^{*\gamma} \quad (65)$$

where $k = \gamma^\gamma(1 - \gamma)^{1-\gamma}$. It is easy to show that the non-cooperative solution is

$$\bar{Y}_{NC}^{\sigma+\phi} = (1 - \gamma)k^{1-\sigma} \left(\frac{\gamma}{1 - \gamma} \right)^{\frac{\gamma(1-\sigma)}{1+\phi}} \quad \bar{Y}_{NC}^{*\sigma+\phi} = \gamma k^{1-\sigma} \left(\frac{1 - \gamma}{\gamma} \right)^{\frac{(1-\gamma)(1-\sigma)}{1+\phi}} \quad (66)$$

By analogy with the single country case, we regard these levels of output as the efficient levels under non-cooperation.

6.2 Choice of Subsidy to Match the Efficient Level

Consider now the decentralized equilibrium, with flexible prices and wages. The home planner maximizes (64), and the foreign planner maximizes (65) with respect to the subsidy rates τ, τ^* subject to the competitive choice of wages and prices. By analogy with the single country case we may write down the following first-order conditions for the latter:

$$\begin{aligned} \frac{W}{P} &= 1/(1-1/\eta)N^\phi C^\sigma & \frac{W^*}{P^*} &= (1-1/\eta)N^{*\phi} C^{*\sigma} \\ 1-1/\zeta &= \frac{1-\tau}{k} \frac{W}{P} S^\gamma & &= \frac{1-\tau^*}{k} \frac{W^*}{P^*} S^{\gamma-1} \end{aligned} \quad (67)$$

where the terms of trade $S = Y/Y^*$.

The home policymaker maximizes (64) subject to all the constraints. The effective mathematics of this is that each policymaker chooses Y, Y^* in the same way as did the social planner above, and then sets the tax rates to achieve this. This yields the Clarida *et al.* (2002) result:

$$(1-1/\eta)(1-1/\zeta) = (1-\tau)(1-\gamma) = (1-\tau^*)\gamma \quad (68)$$

6.3 LQ Approximation for the Small Distortions Case

Here we assume as before that there is inertia in the decision-making of firms, so that there is a price dispersion effect on utility. As a consequence, we may write the utility function for the home country as

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{Y_t^{1+\phi}}{1+\phi} D_t^{1+\phi} \right] \quad s.t. \quad C_t = k Y_t^{1-\gamma} Y_t^{*\gamma} \quad (69)$$

and for the foreign country as

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{*1-\sigma}}{1-\sigma} - \frac{Y_t^{*1+\phi}}{1+\phi} D_t^{*1+\phi} \right] \quad s.t. \quad C_t^* = k Y_t^{1-\gamma} Y_t^{*\gamma} \quad (70)$$

The constraints associated with the home country's price-setting are given by

$$\begin{aligned} 1 &= \xi \Pi_t^{\zeta-1} + (1-\xi) Q_t^{1-\zeta} & H_t - \beta \xi E_t[\Pi_{t+1}^{\zeta-1} H_{t+1}] &= C_t^{-\sigma} Y_t \left(\frac{Y_t^*}{Y_t} \right)^\gamma \\ Q_t H_t &= \Lambda_t & \Lambda_t - \beta \xi E_t[\Pi_{t+1}^\zeta \Lambda_{t+1}] &= \frac{(1-\tau) Y_t^{1+\phi}}{k\alpha} D_t^\phi \\ & & D_t &= \xi \Pi_t^\zeta D_{t-1} + (1-\xi) Q_t^{-\zeta} \end{aligned} \quad (71)$$

Note the difference between these equations and (31)-(34). The optimal price P_H^0 is set in the home currency, so that Q is defined as P_H^0/P_H , but this means that there is a multiplicative term P_H/P to be accounted for. The latter depends on the terms of trade,

which explains the presence of the term $(Y^*/Y)^\gamma$. Note too, that Π refers to home producer price inflation, not consumer price inflation. In a similar manner, the constraints associated with the foreign country's price-setting are given by

$$\begin{aligned}
1 &= \xi \Pi_t^{*\zeta-1} + (1-\xi) Q_t^{*1-\zeta} & H_t^* - \beta \xi E_t[\Pi_{t+1}^{*\zeta-1} H_{t+1}^*] &= k^{-\sigma} C_t^{*-\sigma} Y_t^* \left(\frac{Y_t}{Y_t^*} \right)^{1-\gamma} \\
Q_t^* H_t^* &= \Lambda_t^* & \Lambda_t^* - \beta \xi E_t[\Pi_{t+1}^{*\zeta} \Lambda_{t+1}^*] &= \frac{(1-\tau^*) Y_t^{*1+\phi}}{k\alpha} D_t^{*\phi} \\
&& D_t^* &= \xi \Pi_t^{*\zeta} D_{t-1}^* + (1-\xi) Q_t^{*-\zeta}
\end{aligned} \tag{72}$$

Suppose now that the tax rates are set so as to achieve the social planners' efficient levels of output when inflation is equal to zero. We can now expand (omitting the second order terms in inflation arising from price dispersion) the time- t utility for the home country as:

$$\begin{aligned}
&\frac{(k Y_t^{1-\gamma} Y_t^{*\gamma})^{1-\sigma}}{1-\sigma} - \frac{Y_t^{1+\phi}}{1+\phi} \simeq \frac{(k \bar{Y}_{NC}^{1-\gamma} \bar{Y}_{NC}^{*\gamma})^{1-\sigma}}{1-\sigma} - \frac{\bar{Y}_{NC}^{1+\phi}}{1+\phi} \\
&+ [(1-\gamma)(k \bar{Y}_{NC}^{1-\gamma} \bar{Y}_{NC}^{*\gamma})^{1-\sigma} - \bar{Y}_{NC}^{1+\phi}] y_t + \gamma (k \bar{Y}_{NC}^{1-\gamma} \bar{Y}_{NC}^{*\gamma})^{1-\sigma} y_t^* - \frac{1}{2} (k \bar{Y}_{NC}^{1-\gamma} \bar{Y}_{NC}^{*\gamma})^{1-\sigma} \left[\right. \\
&\left. (1-\gamma)[(\gamma + \phi + (1-\gamma)\sigma) y_t^2] + 2\gamma(1-\gamma)(\sigma-1) y_t y_t^* + \gamma(1-\gamma + \gamma\sigma) y_t^{*2} \right] \tag{73}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(k \bar{Y}_{NC}^{1-\gamma} \bar{Y}_{NC}^{*\gamma})^{1-\sigma}}{1-\sigma} - \frac{\bar{Y}_{NC}^{1+\phi}}{1+\phi} + \gamma (k \bar{Y}_{NC}^{1-\gamma} \bar{Y}_{NC}^{*\gamma})^{1-\sigma} y_t^* - \frac{1}{2} (k \bar{Y}_{NC}^{1-\gamma} \bar{Y}_{NC}^{*\gamma})^{1-\sigma} \left[\right. \\
&\left. (1-\gamma)[(\gamma + \phi + (1-\gamma)\sigma) y_t^2] + 2\gamma(1-\gamma)(\sigma-1) y_t y_t^* + \gamma(1-\gamma + \gamma\sigma) y_t^{*2} \right] \tag{74}
\end{aligned}$$

This result is striking; the use of the tax rate to shift the system to the efficient rate eliminates the first-order term in y_t , but does not eliminate the term in y_t^* . The converse is of course true for the foreign LQ approximation. This would be fine if we were searching for a Nash equilibrium in output deviations. In this case for the home country, the terms in y^* , y^{*2} can be ignored, as they are under the control of the foreign policymaker. However, if we are searching for a Nash equilibrium in inflation π , π^* , then the latter terms cannot be ignored; the utility function is maximized subject to the set of linearized constraints of (71), so that the term in y^* now matters for optimal policy for the home policymaker.

But this violates the requirement that that an appropriate LQ approximation is purely quadratic in the utility function. We therefore have the following result:

Theorem 5

The small distortions LQ approximation is inappropriate as an approximation to the Nash monetary policy equilibrium when inflation is taken as the instrument.

Whether inflation or the output gap is the proxy for interest rates is rarely made clear in most of the theoretical analysis of monetary policy based on micro-foundations. Even if we do go along with the output gap as the instrument, (73) still represents a deviation from Clarida *et al.* (2002), who conclude that the welfare approximation for the home country only involves home variables. It is clear from (73) that even if we can exclude the y_t^* , y_t^{*2} terms from the home country's decision-making process, there is still a term in $y_t y_t^*$ to account for.

6.4 LQ Approximation for the Large Distortions Case

We now apply the Hamiltonian approach to obtain an LQ Approximation that is not subject to the problems of the previous section. Before doing so, we need to carefully choose the appropriate solution concept for the equilibrium of the noncooperative game. Firstly there is the issue of which instrument to use¹¹ ; here we have the choice of the inflation rate or the output gap, each one acting as a proxy for the true instrument, which is the interest rate. We choose the inflation rate, following Benigno and Benigno (2006). Secondly, there is the choice of equilibrium concept given the instrument, which we choose to be *open-loop Nash*. This means that the home country chooses its inflation rate subject to the set of future inflation rates chosen by the foreign country. This is ideally suited to obtaining an LQ approximation using the Hamiltonian approach, but it is not the only Nash solution. The alternative is *closed-loop Nash*, for which the sequence of foreign inflation rates is known to be dependent on the other system variables, such as output, and this is taken into account by the home policymaker when setting its inflation rate. However the latter concept has a solution that can only be obtained numerically even in the LQ case and, as far as we are aware, has not been characterized for non-linear problems. For reasons of tractability therefore, we therefore utilize the open-loop Nash concept.

The problem for the home country then can be described as one of finding the station-

¹¹This was not an issue for the single policymaker, but does arise in non-cooperative games, as the well-known Bertrand versus Cournot Nash equilibria in the oligopoly game clearly illustrates.

ary points of the Lagrangian

$$\begin{aligned}
L = \sum_{t=0}^{\infty} \beta^t & \left[\frac{(kY_t^{1-\gamma}Y_t^{*\gamma})^{1-\sigma}}{1-\sigma} - \frac{Y_t^{1+\phi}}{1+\phi} D_t^{1+\phi} + \lambda_{1t}(1 - \xi \Pi_t^{\zeta-1} - (1-\xi)Q_t^{1-\zeta}) \right. \\
& + \lambda_{2t}(Q_t H_t - \Lambda_t) + \lambda_{3t}(H_t - \beta \xi \Pi_{t+1}^{\zeta-1} H_{t+1} - k^{-\sigma}(Y_t^{1-\gamma}Y_t^{*\gamma})^{1-\sigma}) \\
& + \lambda_{4t}(\Lambda_t - \beta \xi \Pi_{t+1}^{\zeta} \Lambda_{t+1} - \frac{Y_t^{1+\phi}}{k\alpha} D_t^{\phi}) + \lambda_{5t}(D_t - \xi \Pi_t^{\zeta} D_{t-1} - (1-\xi)Q_t^{-\zeta}) \\
& + \lambda_{1t}^*(1 - \xi \Pi_t^{*\zeta-1} - (1-\xi)Q_t^{*1-\zeta}) \\
& + \lambda_{2t}^*(Q_t^* H_t^* - \Lambda_t^*) + \lambda_{3t}^*(H_t^* - \beta \xi \Pi_{t+1}^{*\zeta-1} H_{t+1}^* - k^{-\sigma}(Y_t^{1-\gamma}Y_t^{*\gamma})^{1-\sigma}) \\
& \left. + \lambda_{4t}^*(\Lambda_t^* - \beta \xi \Pi_{t+1}^{*\zeta} \Lambda_{t+1}^* - \frac{Y_t^{*1+\phi}}{k\alpha} D_t^{*\phi}) + \lambda_{5t}^*(D_t^* - \xi \Pi_t^{*\zeta} D_{t-1}^* - (1-\xi)Q_t^{*- \zeta}) \right] \quad (75)
\end{aligned}$$

with respect to all variables other than Π_t^* . Thus we obtain:

$$\begin{aligned}
\frac{\partial L}{\partial Y_t} &= \frac{(1-\gamma)k^{-\sigma}}{Y_t} (Y_t^{1-\gamma}Y_t^{*\gamma})^{1-\sigma} [k - (\lambda_{3t} + \lambda_{3t}^*)(1-\sigma)] - Y_t^{\phi} D_t^{\phi} \left[D_t + \lambda_{4t} \frac{1+\phi}{k\alpha} \right] = 0 \\
\frac{\partial L}{\partial Y_t^*} &= \frac{\gamma k^{-\sigma}}{Y_t^*} (Y_t^{1-\gamma}Y_t^{*\gamma})^{1-\sigma} [k - (\lambda_{3t} + \lambda_{3t}^*)(1-\sigma)] - Y_t^{\phi} D_t^{*\phi} \left[D_t^* + \lambda_{4t}^* \frac{1+\phi}{k\alpha} \right] = 0 \\
\frac{\partial L}{\partial D_t} &= -Y_t^{1+\phi} D_t^{\phi} - \lambda_{4t} \frac{\phi}{k\alpha} Y_t^{1+\phi} D_t^{\phi-1} + \lambda_{5t} - \beta \xi \lambda_{5,t+1} \Pi_{t+1}^{\zeta} = 0 \\
\frac{\partial L}{\partial D_t^*} &= -\lambda_{4t}^* \frac{\phi}{k\alpha} Y_t^{*1+\phi} D_t^{*\phi-1} + \lambda_{5t}^* - \beta \xi \lambda_{5,t+1}^* \Pi_{t+1}^{*\zeta} \\
\frac{\partial L}{\partial Q_t} &= -\lambda_{1t}(1-\xi)(1-\zeta)Q_t^{-\zeta} + \lambda_{2t}H_t + \zeta(1-\xi)\lambda_{5t}Q_t^{-\zeta-1} = 0 \\
\frac{\partial L}{\partial Q_t^*} &= -\lambda_{1t}^*(1-\xi)(1-\zeta)Q_t^{*- \zeta} + \lambda_{2t}^*H_t^* + \zeta(1-\xi)\lambda_{5t}^*Q_t^{*- \zeta-1} = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial H_t} &= \lambda_{2t}Q_t + \lambda_{3t} - \xi \lambda_{3,t-1} \Pi_t^{\zeta-1} = 0 \\
\frac{\partial L}{\partial H_t^*} &= \lambda_{2t}^*Q_t^* + \lambda_{3t}^* - \xi \lambda_{3,t-1}^* \Pi_t^{*\zeta-1} = 0 \\
\frac{\partial L}{\partial \Lambda_t} &= -\lambda_{2t} + \lambda_{4t} - \xi \lambda_{4,t-1} \Pi_t^{\zeta} = 0 \\
\frac{\partial L}{\partial \Lambda_t^*} &= -\lambda_{2t}^* + \lambda_{4t}^* - \xi \lambda_{4,t-1}^* \Pi_t^{*\zeta} = 0 \\
\frac{\partial L}{\partial \Pi_t} &= (1-\zeta)\xi \lambda_{1t} \Pi_t^{\zeta-2} - \lambda_{3t} \xi (\zeta-1) \Pi_t^{\zeta-2} H_t - \lambda_{4t} \xi \zeta \Pi_t^{\zeta-1} \Lambda_t - \xi \zeta \lambda_{5t} \Pi_t^{\zeta-1} D_{t-1} = 0
\end{aligned}$$

The steady state is given by

$$\begin{aligned}
\Pi = Q = D = 1 = \Pi^* = Q^* = D^* \quad \Lambda = H = \frac{Y^{1+\phi}}{k\alpha(1-\beta\xi)} = \Lambda^* = H^* \\
Y^{\sigma+\phi} = \alpha k^{1-\sigma} = Y^{*\sigma+\phi}
\end{aligned}$$

$$\begin{aligned}
(1 - \gamma - \alpha)k &= (\lambda_3 + \lambda_3^*)(1 - \sigma)\gamma - \lambda_3(1 + \phi) & \gamma k &= (\lambda_3 + \lambda_3^*)(1 - \sigma)\gamma - \lambda_3^*(1 + \phi) \\
Y^{1+\phi}(1 - \lambda_3 \frac{\phi}{k\alpha}) &= (1 - \beta\xi)\lambda_5 & -Y^{1+\phi}\lambda_3^* \frac{\phi}{k\alpha} &= (1 - \beta\xi)\lambda_5^* \\
-\lambda_1(1 - \xi)(1 - \zeta) + \lambda_2 H + \lambda_5 \zeta(1 - \xi) &= 0 & -\lambda_1^*(1 - \xi)(1 - \zeta) + \lambda_2^* H^* + \lambda_5^* \zeta(1 - \xi) &= 0 \\
\lambda_4 &= -\lambda_3 & \lambda_4^* &= -\lambda_3^* & \lambda_2 &= -\lambda_3(1 - \xi) & \lambda_2^* &= -\lambda_3^*(1 - \xi)
\end{aligned}$$

We can now expand (75) about its steady state in order to evaluate the second-order approximation in the welfare, using the steady state values of the Lagrange multipliers $\lambda_1, \dots, \lambda_5^*$. Defining $y_t = (Y_t - Y)/Y$, $\pi_t = \Pi_t - 1$, $y_t^* = (Y_t^* - Y)/Y$, $\pi_t^* = \Pi_t^* - 1$, we first note that it is easy to show that $\partial L/\partial \Pi^* = 0$, so that the coefficient of π_t^* is zero. After further manipulation we can show that the period- t welfare is given by:

$$\begin{aligned}
u_t = -\frac{Y^{1+\phi}}{2} \frac{\phi + 1 - \alpha + \alpha\sigma}{\sigma + \phi} & \left[(\sigma + \phi)((1 - \gamma)y_t + \gamma y_t^*)^2 + \gamma(1 - \gamma)(1 + \phi)(y_t - y_t^*)^2 \right. \\
& \left. + \frac{\xi\zeta}{(1 - \xi)(1 - \beta\xi)} ((1 - \gamma)\pi_t^2 + \gamma\pi_t^{*2}) \right] \tag{76}
\end{aligned}$$

For further details see Appendix C. The first term in this expression represents consumption deviations and the second, terms of trade deviations. The presence of foreign country inflation deviations arises from the fact that they are associated with foreign country price dispersion. The latter enters the foreign price-setting decision, which in turn has an impact on the home country's consumption choice.

What is immediately noticeable in (76) is the symmetry with which the terms in output and inflation deviations from each bloc enter the quadratic approximation to the utility of the home bloc. It implies of course that it is identical to the approximation to the utility of the foreign bloc. Furthermore, a similar calculation for the cooperative utility function produces an identical approximation. Thus we are able to replicate the Benigno and Benigno (2006) result in the absence of shocks, that for identical economies when the system starts out of equilibrium, there are no gains from cooperation.

In the presence of shocks, the structure of the welfare approximation would be identical to (76) apart from there now being output deviation targets that would depend on the shocks. These are obtained via calculation of the second derivative terms $\partial^2 L/\partial Y \partial S$, where S represents a shock. As in Benigno and Benigno (2006), for some shocks the output targets are different under cooperation and non-cooperation, so that there are potential gains from cooperation even for identical economies.

7 Concluding Remarks

Despite recent advances in numerical methods for non-linear optimization problems, the ‘curse of dimensionality’ (see Judd (1998), chapter 7) will ensure the usefulness of LQ approximations to these problem, even for the case of a single policy-maker. For games involving many policy-makers a LQ approximation to calculations of players is for some equilibrium concepts, for example the closed-loop Nash equilibrium, probably essential. This paper has attempted a ‘users’ guide’ to accurate LQ approximation for researchers studying such problems. We have highlighted pitfalls already exposed in the literature and added another one in emphasizing that second-order conditions, usually ignored in the optimal policy literature, do matter.

The Hamiltonian method of Magill (1977), which we have shown is equivalent to the Benigno-Woodford ‘large distortions’ procedure, provides an accurate LQ approximation of the household’s utility function given a linearized model economy *in the vicinity of the Ramsey commitment problem for the policymaker*. For the case of non-cooperative games, the latter is the Ramsey problem for each policy-maker given the open-loop trajectory of instruments of the other players. The question then is, given the choice of welfare which differs for cooperative and non-cooperative games, is this LQ approximation appropriate for other types of policy (for example for optimized simple rules, time-consistent policy or for other non-cooperative equilibrium concepts)? Because the Ramsey commitment problem is, ex ante, the best the policymaker can achieve, as Woodford (2003), chapter 6, has pointed out, this is indeed the case.¹² Thus LQ approximation provides a tractable framework for comparing both cooperative and non-cooperative rules with and without commitment, and different forms of non-cooperative equilibria (closed-loop versus open-loop, different choices of instruments) using the *same* LQ approximation of the problem facing each policymaker. Future research involving the authors will pursue precisely this agenda.

¹²He writes that “... this calculation (of the quadratic approximation) need only be done once, and not separately for each type of policy that one may wish to study”.

A The Equivalence of the Benigno-Woodford and Hamiltonian Procedures

A.1 The Benigno-Woodford Procedure

To find $(\bar{\pi}, \bar{x})$ and h first write

$$\begin{aligned} (x_t - x^*)^2 + \pi_t^2 &\equiv (x_t - \bar{x} + \bar{x} - x^*)^2 + (\pi_t - \bar{\pi} + \bar{\pi})^2 \\ &= (x_t - \bar{x})^2 + 2(x_t - \bar{x})(\bar{x} - x^*) + (\pi_t - \bar{\pi})^2 + 2\bar{\pi}(\pi_t - \bar{\pi}) \\ &\quad + \text{constant terms} \end{aligned} \tag{A.1}$$

Then (11) holds iff at each time t

$$\begin{aligned} \theta(x_t - \bar{x})^2 + \phi(\pi_t - \bar{\pi})^2 &\equiv (x_t - \bar{x})^2 + 2(x_t - \bar{x})(\bar{x} - x^*) + (\pi_t - \bar{\pi})^2 + 2\bar{\pi}(\pi_t - \bar{\pi}) \\ &\quad + h \left(\frac{\beta}{\lambda}(\pi_t - \bar{\pi}) - (\pi_t - \bar{\pi}) - a(x_t - \bar{x}) + b(x_t - \bar{x})^2 \right) \end{aligned} \tag{A.2}$$

Equating quadratic and linear terms we arrive at

$$\theta = 1 + hb \tag{A.3}$$

$$\phi = 1 \tag{A.4}$$

$$2(\bar{x} - x^*) - ha = 0 \tag{A.5}$$

$$2\bar{\pi} + h \left(\frac{\beta}{\lambda} - 1 \right) = 0 \tag{A.6}$$

Then together with the condition for $(\bar{\pi}, \bar{x})$ to be a steady state:

$$(\beta - 1)\bar{\pi} - f(\bar{x}) = 0 \tag{A.7}$$

we have 5 equations to solve to $\theta, \phi, h, \bar{\pi}$ and \bar{x} . The solution is

$$h = -\frac{2(x^* - \bar{x})}{a} < 0 \text{ if } x^* > \bar{x} \tag{A.8}$$

$$\bar{\pi} = \left(1 - \frac{\beta}{\lambda}\right) \frac{h}{2} > 0 \text{ iff } \beta > \lambda \text{ and } x^* > \bar{x} \tag{A.9}$$

$$\theta = 1 + hb < 1 \text{ if } x^* > \bar{x} \tag{A.10}$$

where \bar{x} is the solution to

$$(\beta - 1) \left(1 - \frac{\beta}{\lambda}\right) \frac{(x^* - \bar{x})}{a} + f(\bar{x}) = 0 \tag{A.11}$$

A.2 The Hamiltonian Procedure

We now show that the LQ procedure of BW is equivalent to a rather simpler one. Consider the deterministic problem to choose at $t = 0$ a trajectory $\{\pi_t\}$ so as to minimize

$$\left[\sum_{t=0}^{\infty} \lambda^t [(x_t - x^*)^2 + \pi_t^2] \right] \quad (\text{A.12})$$

subject to

$$\pi_t = \beta\pi_{t+1} + f(x_t) \quad (\text{A.13})$$

To solve this problem we minimize a Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \lambda^t [(x_t - x^*)^2 + \pi_t^2 + \mu_t(\beta\pi_{t+1} - \pi_t + f(x_t))] \quad (\text{A.14})$$

with respect to $\{\pi_t\}$, $\{x_t\}$ and the Lagrangian multiplier $\{\mu_t\}$. This is the deterministic component of our original non-linear optimization problem available to the policymaker if she can commit. The first-order conditions for this problem are

$$2(x_t - x^*) + \mu_t f'(x_t) = 0 \quad (\text{A.15})$$

$$2\pi_t - \mu_t + \frac{\beta}{\lambda} \mu_{t-1} = 0 \quad (\text{A.16})$$

$$\pi_t - \beta\pi_{t+1} - f(x_t) = 0 \quad (\text{A.17})$$

This system has a steady state (x, π, μ) at

$$2(x - x^*) + \mu f'(x) = 0 \quad (\text{A.18})$$

$$2\pi - \mu \left(1 - \frac{\beta}{\lambda}\right) = 0 \quad (\text{A.19})$$

$$(1 - \beta)\pi - f(x) = 0 \quad (\text{A.20})$$

Comparing (A.18) to (A.20) with (A.7) to (A.9) and noting that $a = f'(\bar{x})$ in (A.5) it is immediately apparent that $(x, \pi, \mu) = (\bar{x}, \bar{\pi}, h)$ found in the BW procedure. Then the modified loss function (11) is a second-order Taylor series approximation to the Lagrangian (A.14) evaluated at the steady state of the optimal commitment solution in the vicinity of (x, π, μ) .

B Proof of Proposition

(i) Firstly, we require the coefficient of π_t^2 inside the brackets of (53), $\alpha + (1 + \phi)\lambda_5$, to be positive. A little calculation shows that (with $\alpha > 1 - h_C$) this term is greater than $1 - h_C$ provided that $\sigma > 1$. Ignoring the shock term a_t , if we now consider the

remaining terms as a quadratic function of y_t and $y_t - z_t$, then this quadratic will always be positive provided that (a) $\alpha + (1 + \phi)\lambda_5 > 0$, (b) $\frac{\sigma}{1-h_C}(1 - \frac{\lambda_5(1+\sigma)}{1-h_C}) > 0$ and (c) $\phi(\alpha + (1 + \phi)\lambda_5)\frac{\sigma}{1-h_C}(1 - \frac{\lambda_5(1+\sigma)}{1-h_C}) - \frac{\lambda_5^2\sigma^2}{(1-h_C)^2} > 0$. (a) has already been shown, and since by assumption $\alpha > 1 - h_C$ it follows that $\lambda_5 < 0$, so the left hand side of (b) is greater 0. After some manipulation we can show that after multiplying (c) through by $(1 - h_C)^2$ the left hand side becomes

$$\begin{aligned} & (\phi^3 - \phi - \sigma)(1 - h_C)^2 + 2\alpha(1 - h_C)(\sigma\phi^2 + \phi + \sigma) + (\phi\sigma^2 - \phi - \sigma)\alpha^2 \\ & > (\phi^3 + \sigma\phi^2)(1 - h_C)^2 + \alpha(1 - h_C)(\sigma\phi^2 + \phi + \sigma) + (\phi\sigma^2 - \phi - \sigma)\alpha^2 \end{aligned} \quad (\text{B.21})$$

where the inequality holds when $\alpha > 1 - h_C$. Thus the sufficient condition $\phi\sigma^2 - \phi - \sigma > 0$ is likely to be considerably more stringent a condition than is required.

(ii) With $\alpha < 1 - h_C$, it is clear that $\lambda_5 > 0$, so that (a) above is satisfied. After a little manipulation, it is easy to show that $\frac{\sigma}{1-h_C}(1 - \frac{\lambda_5(1+\sigma)}{1-h_C}) = \frac{\sigma}{(1-h_C)^2(\sigma+\phi)}(\alpha(1 + \sigma) - (1 - h_C)(1 - \phi))$, so that (b) is satisfied if $\alpha(1 + \sigma) > (1 - h_C)(1 - \phi)$. Finally, using the condition $\alpha < 1 - h_C$,

$$\begin{aligned} & (\phi^3 - \phi - \sigma)(1 - h_C)^2 + 2\alpha(1 - h_C)(\sigma\phi^2 + \phi + \sigma) + (\phi\sigma^2 - \phi - \sigma)\alpha^2 \\ & > (\phi^3 - \sigma - \phi)(1 - h_C)^2 + \alpha(1 - h_C)(\sigma\phi^2 + \phi + \sigma) + (\phi\sigma^2 + \sigma\phi^2)\alpha^2 \end{aligned} \quad (\text{B.22})$$

where the inequality holds when $\alpha < 1 - h_C$. Once again the sufficient condition $\phi^3 - \phi - \sigma > 0$ is likely to be considerably more stringent a condition than is required.

C Derivation of Second-Order Welfare Approximation in the Two-Bloc Case:

From now on, we drop all t -subscripts for purposes of conciseness. Firstly, we note that

$$\frac{\partial L}{\partial \Pi^*} = -\xi(\zeta - 1)\Pi^{*\zeta-1}\lambda_1^* - \xi(\zeta - 1)\Pi^{*\zeta-2}H^*\lambda_3^* - \xi\zeta\Pi^{*\zeta-1}\Lambda^*\lambda_4^* - \xi\zeta\Pi^{*\zeta-1}D\lambda_5^* \quad (\text{C.23})$$

which is easily shown to equal 0. Next, the coefficient of $\frac{1}{2}y_t^2$ is given by

$$\begin{aligned} \frac{1}{Y^2} \frac{\partial^2 L}{\partial Y^2} &= \frac{Y^{1+\phi}}{\alpha} \left[\left(1 - \frac{\lambda_3 + \lambda_3^*}{k}(1 - \sigma)\right)(1 - \gamma)((1 - \sigma)(1 - \gamma) - 1) - \phi\alpha - \frac{\lambda_4}{k}\phi(1 + \phi) \right] \\ &= -\frac{Y^{1+\phi}}{\alpha} \frac{\phi + 1 - \alpha + \alpha\sigma}{\phi + \sigma} (1 - \gamma)(\phi + \sigma - \sigma\gamma + \gamma) \end{aligned} \quad (\text{C.24})$$

that of $y_t y_t^*$ is given by

$$\begin{aligned} \frac{1}{YY^*} \frac{\partial^2 L}{\partial Y \partial Y^*} &= \frac{Y^{1+\phi}}{\alpha} (1 - \gamma)\gamma(1 - \sigma) \left(1 - \frac{\lambda_3 + \lambda_3^*}{k}(1 - \sigma)\right) \\ &= -\frac{Y^{1+\phi}}{\alpha} \frac{\phi + 1 - \alpha + \alpha\sigma}{\phi + \sigma} (1 - \gamma)\gamma(1 - \sigma) \end{aligned} \quad (\text{C.25})$$

and that of $\frac{1}{2}y_t^{*2}$ is given by

$$\begin{aligned} \frac{1}{Y^2} \frac{\partial^2 L}{\partial Y^2} &= \frac{Y^{1+\phi}}{\alpha} \left[\left(1 - \frac{\lambda_3 + \lambda_3^*}{k} (1 - \sigma)\right) \gamma ((1 - \sigma)\gamma - 1) + \frac{\lambda_3^*}{k} \phi (1 + \phi) \right] \\ &= -\frac{Y^{1+\phi}}{\alpha} \frac{\phi + 1 - \alpha + \alpha\sigma}{\phi + \sigma} \gamma (\phi + \gamma\sigma + 1 - \gamma) \end{aligned} \quad (\text{C.26})$$

Summing these yields the first two terms in (76), expressed as deviations in total output and the terms of trade. The second-order terms in π_t^2 are derived from the sum of terms in $\frac{\partial^2 L}{\partial \Pi^2}$, $2 \frac{\partial^2 L}{\partial \Pi \partial H}$, $2 \frac{\partial^2 L}{\partial \Pi \partial \Lambda}$, $2 \frac{\partial^2 L}{\partial \Pi \partial H}$, $\frac{\partial^2 L}{\partial Q^2}$:

$$\begin{aligned} &-\frac{1}{2} \xi (\zeta - 1) \pi_t^2 [(\zeta - 2)\lambda_1 + (\zeta - 2)\lambda_3 H + \zeta \lambda_4 H + \zeta \lambda_5] - \pi_t h_t \xi (\zeta - 1) \lambda_3 \\ &\quad - \pi_t \lambda_t \zeta \lambda_4 + q_t h_t \lambda_2 + \frac{1}{2} q_t^2 (1 - \xi) \zeta [(1 - \zeta)\lambda_1 - (\zeta + 1)\lambda_5] \end{aligned} \quad (\text{C.27})$$

where $h_t = H_t - H$, $\lambda_t = \Lambda_t - \Lambda$, $q = Q_t - 1$. Substituting $h_t + Hq_t = \lambda_t$, $\xi \pi_t = (1 - \xi)q_t$ yields the required term of (76). The coefficient of $\frac{1}{2}\pi_t^{*2}$ is given by

$$\frac{\partial^2 L}{\partial \Pi^{*2}} = -\xi (\zeta - 1) [(\zeta - 2)\lambda_1^* + (\zeta - 2)\lambda_3^* H + \zeta \lambda_4^* H + \zeta \lambda_5^*] \quad (\text{C.28})$$

Finally, it is easy to show that all partial second derivatives $\partial^2 L / \partial \Pi^i \partial Y^i = 0$ (where $\Pi^i = \Pi, \Pi^*$, $Y^i = Y, Y^*$), so that there are no cross-terms of the form $\pi_t y_t$ in the approximation.

References

- Benigno, G. and Benigno, P. (2006). Designing Targeting Rules for International Monetary Cooperation. *Journal of Monetary Economics*, to appear.
- Benigno, P. and Woodford, M. (2003). Optimal Monetary and Fiscal policy: A Linear-Quadratic Approach. Presented at the International Research Forum on Monetary Policy in Washington, DC, November 14-15, 2003.
- Benigno, P. and Woodford, M. (2005). Inflation Stabilization and Welfare: The Case of a Distorted Steady State. *Journal of the European Economic Association*, **3**(6), 1–52.
- Choudhary, A. and Levine, P. (2006). Idle Worship. *Economic Letters*, **90**(1), 77–83.
- Clarida, R., Galí, J., and Gertler, M. (1999). The Science of Monetary Policy: A New Keynesian Perspective. *Journal of Economic Literature*, **37**(4), 1661–1707.
- Clarida, R., Gali, J., and Gertler, M. (2000). Monetary policy rules and macroeconomic stability: Evidence and some theory. *Quarterly Journal of Economics*, **115**, 147–180.

- Clarida, R., Gali, J., and Gertler, M. (2002). A simple framework for international monetary policy analysis. *Journal of Monetary Economics*, **49**, 879–904.
- Currie, D. and Levine, P. (1993). *Rules, Reputation and Macroeconomic Policy Coordination*. CUP.
- Judd, K. L. (1998). *Numerical Methods in Economics*. The MIT Press, Cambridge, MA, USA.
- Khan, A., King, R., and Wolman, A. (2003). Optimal Monetary Policy. *Review of Economic Studies*, **70**, 825–860.
- Kim, J. and Kim, H. K. (2003). Spurious Welfare Reversals in International Business Cycle Models. *Journal of International Economics*, **60**, 471–500.
- Kim, J. and Kim, H. K. (2006). Two Pitfalls of Linearization Methods. *Journal of Money, Credit and Banking*. Forthcoming.
- Kim, J., Levin, A., and Yun, T. (2006a). Diagnosing and Treating Bifurcations in Perturbation Analysis of Dynamic Macro Models. Forthcoming IMF Working Paper.
- Kim, J., Levin, A., and Yun, T. (2006b). Relative Price Distortion and Optimal Monetary Policy in Open Economies. *Bank of Korea Working Paper 251*.
- Levine, P., McAdam, P., and Pearlman, J. (2006). Quantifying and Sustaining the Welfare Gains from Monetary Commitment. Forthcoming ECB Working Paper, presented at the 12th International Conference on Computing in Economics and Finance, Cyprus, June, 2006.
- Magill, M. (1977). A Local Analysis of Capital Accumulation under Uncertainty. *Journal of Economic Theory*, **15**(2), 211–219.
- McCallum, N. T. and Nelson, E. (2004). Targeting Rules vs Instrument Rules for Monetary Policy. Mimeo.
- Smets, F. and Wouters, R. (2003). An Estimated Stochastic Dynamic General Equilibrium Model of the Euro Area. *Journal of the European Economic Association*, **1**(5), 1123–1175.
- Svensson, L. E. O. (2003). What is Wrong with Taylor Rules? Using Judgement in Monetary Policy through Targeting Rules. *Journal of Economic Literature*, **41**, 426–77.

Svensson, L. E. O. (2005). Monetary Policy with Judgment: Forecast Targeting. *International Journal of Central Banking*, **1**, 1–54.

Woodford, M. (2003). *Foundations of a Theory of Monetary Policy*. Princeton University Press.